Lectures on Dynamical Systems

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Part 1

LECTURE 1

Introduction

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The name of the subject, "DYNAMICAL SYSTEMS", came from the title of classical book: G.D.Birkhoff, Dynamical Systems. Amer. Math. Soc. Colloq. Publ. 9. American Mathematical Society, New York (1927), 295 pp.

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II. *Time* can be either discrete, whose set of values is the set of integer numbers $\mathbb{Z}$, or continuous, whose set of values is the set of real numbers $\mathbb{R}$.
III. Law of evolution is the rule which allows us, if we know the state of the system at some moment of time, to determine the state of the system at any other moment of time. (The existence of this law is equivalent to the assumption that our process is deterministic in the past and in the future.)

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It is assumed that the law of evolution itself does not depend on time, i.e for any values t, t_0 the result of the evolution during the time t starting from the moment of time t_0 does not depend on t_0.

Definition of dynamical system, continued

Denote X the phase space of our system. Let us introduce *the evolution operator g^t for the time t* by means of the following relation: for any state $x \in X$ of the system at the moment of time 0 the state of the system at the moment of time t is $g^t x$.
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Therefore, the set $\{g^t\}$ is commutative group with respect to the composition operation: $g^s g^t = g^s(g^t)$.
Unity of this group is g^0 which is the identity transformation.
The inverse element to g^t is g^{-t}.

This group is isomorphic to $\mathbb{Z}$ or $\mathbb{R}$ for the cases of discrete or continuous time respectively.

In the case of discrete time, such groups are called one-parametric groups of transformations with discrete time, or *phase cascades*.

In the case of continuous time, such groups are just called one-parametric groups of transformations, or *phase flows*.

Definition of dynamical system, continued

Now we can give a formal definition.

Definition

Dynamical system is a triple (X, Ξ, G), where X is a set (*phase space*), Ξ is either $\mathbb{Z}$ or $\mathbb{R}$, and G is a one-parametric group of transformation of X (with discrete time if $\Xi = \mathbb{Z}$).

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Remark

For the case of discrete time $g^n = (g^1)^n$. So, the orbit of the point x is $\ldots, x, g^1 x, (g^1)^2 x, (g^1)^3 x, \ldots$

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In our course we almost always will have a finite-dimensional smooth manifold as X and will assume that $g^t x$ is smooth with respect to x if $\Xi = \mathbb{Z}$ and with respect to (x, t) if $\Xi = \mathbb{R}$. So, we consider *smooth dynamical systems*.

Simple examples

Example (Circle rotation)

$X = \mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$, $\Xi = \mathbb{Z}$, $g^1 : \mathbb{S}^1 \to \mathbb{S}^1$, $g^1 x = x + \alpha \mod 2\pi, \alpha \in \mathbb{R}$.

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Example (Torus winding)

$X = \mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, $\Xi = \mathbb{R}$, $g^t : \mathbb{T}^2 \to \mathbb{T}^2$,

$$g^t \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + t\omega_1 \mod 2\pi \\ x_2 + t\omega_2 \mod 2\pi \end{pmatrix}$$

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Example (Exponent)

$X = \mathbb{R}, \Xi = \mathbb{R}$, $g^t : \mathbb{R} \to \mathbb{R}$, $g^t x = e^t x$.

More examples

Example (Fibonacci sequence)

$b_{k+1} = b_k + b_{k-1},\ k = 1, 2, \ldots;\ b_0 = 0,\ b_1 = 1$

Denote $x_k = \begin{pmatrix} b_{k-1} \\ b_k \end{pmatrix},\ A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$

Then $x_{k+1} = Ax_k$. Therefore $x_{k+1} = A^k x_1$.

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In this example $X = \mathbb{R}^2$, $\Xi = \mathbb{Z}$.
Characteristic equation is $\det \begin{pmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = 0$, or $\lambda^2 - \lambda - 1 = 0$.

Eigenvalues are $\lambda_{1,2} = \frac{1}{2}(1 \pm \sqrt{5})$. Eigenvectors are $\xi_{1,2} = \begin{pmatrix} 1 \\ \lambda_{1,2} \end{pmatrix}$.

If $x_1 = c_1\xi_1 + c_2\xi_2$, then $x_{k+1} = c_1\lambda_1^k\xi_1 + c_2\lambda_2^k\xi_2$.
From initial data $c_1 = -c_2 = 1/(\lambda_1 - \lambda_2) = 1/\sqrt{5}$.

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Example (Two body problem)

$X = \mathbb{R}^{12},\ \Xi = \mathbb{R}$

Let $(X, \mathbb{R}, G)$ be a smooth dynamical system, $G = \{g^t, t \in \mathbb{R}\}$. It defines a vector field v on X :

$$v(x) = (\frac{d}{dt} g^t x)_{t=0}$$

This vector field defines an autonomous ODE

$$\frac{dx}{dt} = v(x)$$

Then $g^t x, t \in \mathbb{R}$ is the solution to this ODE with the initial condition x at $t = 0$. Indeed,

$$\frac{d}{dt} g^t x = (\frac{d}{d\varepsilon} g^{t+\varepsilon} x)_{\varepsilon=0} = (\frac{d}{d\varepsilon} g^{\varepsilon} g^t x)_{\varepsilon=0} = v(g^t x)$$

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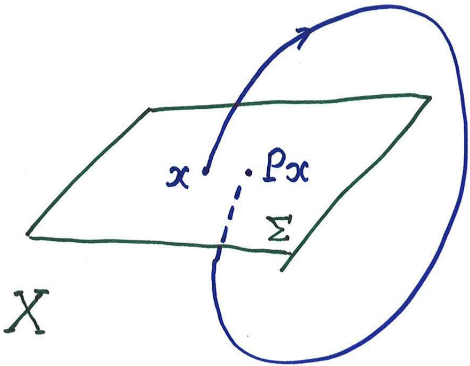
Dynamical systems with continuous time are usually described via corresponding autonomous ODEs.



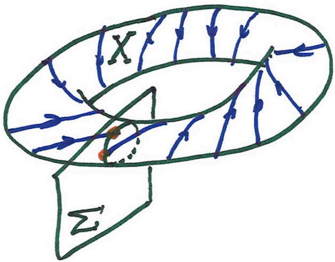
$P : \Sigma \to \Sigma$

Σ is called *a Poincaré surface of section*.

P is called *a Poincaré first return map*. It generates a new dynamical system with discrete time.

Example

Section of torus



For this surface of section the Poincaré first return map for a torus winding is a rotation of the circle.

A non-autonomous ODE

$$\frac{dx}{dt} = v(x, t)$$

can be reduced to an autonomous one by introducing a new dependent variable y: $dy/dt = 1$. However, this is often an inappropriate approach because the recurrence properties of the time dependence are thus hidden.

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Example (Quasi-periodic time dependence)

$$\frac{dx}{dt} = v(x, t\omega), x \in \mathbb{R}^n, \omega \in \mathbb{R}^m,$$

function v is 2π-periodic in each of the last m arguments. It is useful to study the autonomous ODE

$$\frac{dx}{dt} = v(x, \varphi), \; \frac{d\varphi}{dt} = \omega$$

whose phase space is $\mathbb{R}^n \times \mathbb{T}^m$. For $m = 1$ the Poincaré return map for the section $\varphi = 0\, mod\, 2\pi$ reduces the problem to a dynamical system with discrete time.

Blow-up

For ODEs some solutions may be defined only locally in time, for $t_- < t < t_+$, where t_-, t_+ depend on initial condition. An important example of such a behavior is a "blow-up", when a solution of a continuous-time system in $X = \mathbb{R}^n$ approaches infinity within a finite time.

Example

For equation

$$\dot{x} = x^2,\ x \in \mathbb{R}$$

each solution with a positive (respectively, a negative) initial condition at $t = 0$ tends to $+\infty$ (respectively, $-\infty$) when time approaches some finite moment in the future (respectively, in the past). The only solution defined for all times is $x \equiv 0$.

Such equations define only *local phase flows*.

Some generalisations

1. One can modify the definition of dynamical system taking $\Xi = \mathbb{Z}^+$ or $\Xi = \mathbb{R}^+$, and G being semigroup of transformations.

2. There are theories in which phase space is an infinite-dimensional functional space. (However, even in these theories very often essential events occur in a finite-dimensional submanifold, and so the finite-dimensional case is at the core of the problem. Moreover, analysis of infinite-dimensional problems often follows the schemes developed for finite-dimensional problems.)

Topics in the course

1. Linear dynamical systems.
2. Normal forms of nonlinear systems.
3. Bifurcations.
4. Perturbations of integrable systems, in particular, KAM-theory.

Exercises

1. Consider the sequence

$$1,\ 2,\ 4,\ 8,\ 1,\ 3,\ 6,\ 1,\ 2,\ 5,\ 1,\ 2\ ,4,\ 8, \ldots$$

of first digits of consecutive powers of 2. Does a 7 ever appears in this sequence? More generally, does 2^n begin with an arbitrary combination of digits?

2. Prove that $\sup_{0<t<\infty} (\cos t + \sin \sqrt{2}t) = 2$.

LECTURE 2

LINEAR DYNAMICAL SYSTEMS

Consider ODE

$$\dot{x} = v(t,x),\ \ t \in \mathbb{R},\ \ x \in D \subset \mathbb{R}^n,\ \ v(\cdot,\cdot) \in C^2(\mathbb{R} \times D)$$

Let $x_*(t)$, $t \in \mathbb{R}$ be a solution to this equation.
Introduce $\xi = x - x_*(t)$. Then

$$\dot{\xi} = v(t, x_*(t) + \xi) - v(t, x_*(t)) = \frac{\partial v(t, x_*(t))}{\partial x}\xi + O(|\xi|^2)$$

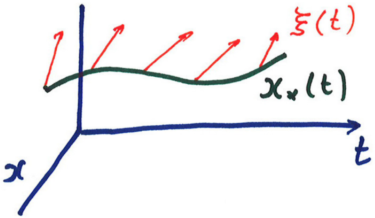
Denote $A(t) = \frac{\partial v(t, x_*(t))}{\partial x}$

Definition

Linear non-autonomous ODE

$$\dot{\xi} = A(t)\xi$$

is called *the variation equation* near solution $x_*(t)$.



Consider a linear (homogeneous) non-autonomous ODE

$$\dot{x} = A(t)x, \; x \in \mathbb{R}^n$$

where $A(t)$ is a linear operator, $A(t)\colon \mathbb{R}^n \to \mathbb{R}^n$, $A(\cdot) \in C^0(\mathbb{R})$.

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In a fixed basis in $\mathbb{R}^n$ one can identify the vector x with the column of its coordinates in this basis and the operator $A(t)$ with its matrix in this basis:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{pmatrix}$$

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Then the equation takes the form:

$$\begin{aligned} \dot{x}_1 &= a_{11}(t)x_1 + a_{12}(t)x_2 + \ldots + a_{1n}(t)x_n \\ \dot{x}_2 &= a_{21}(t)x_1 + a_{22}(t)x_2 + \ldots + a_{2n}(t)x_n \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dot{x}_n &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \ldots + a_{nn}(t)x_n \end{aligned}$$

In this form the equation is called "a system of n homogeneous linear non-autonomous differential equations of the first order".

Theorem

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Theorem

The set of all solutions of a linear non-autonomous ODE in $\mathbb{R}^n$ *is a linear n-dimensional space.*

Proof.

The set of solutions is isomorphic to the phase space, i.e. to $\mathbb{R}^n$. An isomorphism maps a solution to its initial (say, at $t = 0$) datum. □

Definition

Any basis in the space of solutions is called *a fundamental system of solutions*.

Linear constant-coefficient ODEs

Consider an ODE

$$\dot{x} = Ax,\ x \in \mathbb{R}^n,$$

where A is a linear operator, $A : \mathbb{R}^n \to \mathbb{R}^n$.
Denote $\{g^t\}$ the phase flow associated with this equation.

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Definition

The exponent of the operator tA is the linear operator $e^{tA} : \mathbb{R}^n \to \mathbb{R}^n$:

$$e^{tA} = E + tA + \tfrac{1}{2}(tA)^2 + \frac{1}{3!}(tA)^3 + \dots,$$

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Theorem
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Proof.
$\frac{d}{dt} e^{tA} x = Ae^{tA} x,\ \ e^{0\cdot A} x = x$ □

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Theorem

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Proof.

$\frac{d}{dt} e^{tA}x = Ae^{tA}x, \quad e^{0 \cdot A}x = x$ □

The eigenvalues of A are roots of the characteristic equation: $\det(A - \lambda E) = 0$.
If there are complex eigenvalues, then it is useful to *complexify* the problem.

Complexified equation:

$$\dot{z} = Az,\ z \in \mathbb{C}^n,$$

and now $A \colon \mathbb{C}^n \to \mathbb{C}^n,\ A(x + iy) = Ax + iAy$.

Linear constant-coefficient ODEs, continued

Let V be an l-dimensional linear vector space over $\mathbb{C}$, B be a linear operator, $B : V \to V$.

Definition

Operator B is called *a Jordan block* with eigenvalue λ, if its matrix in a certain basis (called *the Jordan basis*) is the Jordan block:

$$\begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & \dots & 0 & \lambda \end{pmatrix}$$

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If B is a Jordan block with eigenvalue λ, then the matrix of the exponent of tB in this basis is

$$e^{\lambda t} \begin{pmatrix} 1 & t & t^2/2 & \dots & t^{l-1}/(l-1)! \\ 0 & 1 & t & \dots & t^{l-2}/(l-2)! \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & t \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Theorem (The Jordan normal form)

Space $\mathbb{C}^n$ decomposes into a direct sum of invariant with respect to A and e^{tA} subspaces, $\mathbb{C}^n = V_1 \oplus V_2 \oplus \ldots \oplus V_m$, such that in each of this subspaces A acts as a Jordan block. For V_j such that the eigenvalue of the corresponding Jordan block is not real, both V_j and its complex conjugate $\bar{V}_j$ are presented in this decomposition.

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De-complexification

Note that $V_j \oplus \bar{V}_j = \mathrm{Re}\, V_j \oplus \mathrm{Im}\, V_j$ over $\mathbb{C}$.
Thus, $\mathbb{C}^n = V_1 \oplus \ldots \oplus V_r \oplus (\mathrm{Re}\, V_{r+1} \oplus \mathrm{Im}\, V_{r+1}) \oplus \ldots \oplus (\mathrm{Re}\, V_k \oplus \mathrm{Im}\, V_k)$ if the field of the coefficients is $\mathbb{C}$. In this decompositions subspaces $V_1, \ldots, V_r$ correspond to real eigenvalues, and subspaces $V_{r+1}, \ldots, V_k$ correspond to complex eigenvalues, $r + 2k = m$.

Now

$$\mathbb{R}^n = \mathrm{Re} V_1 \oplus \ldots \oplus \mathrm{Re} V_r \oplus (\mathrm{Re}\, V_{r+1} \oplus \mathrm{Im}\, V_{r+1}) \oplus \ldots \oplus (\mathrm{Re}\, V_k \oplus \mathrm{Im}\, V_k)$$

over the field of the coefficients $\mathbb{R}$.

Thus we can calculate $e^{tA}x$ for any $x \in \mathbb{R}^n$.

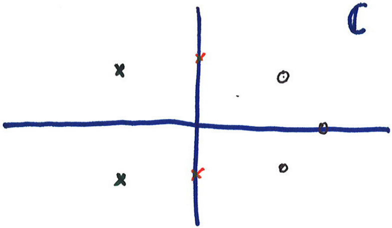
Linear constant-coefficient ODEs, stability

Definition

A linear ODE is *stable* if all its solutions are bounded.
A linear ODE is *asymptotically stable* if all its solutions tend to 0 as $t \longrightarrow +\infty$.

Theorem

A linear constant-coefficient ODE is
a) stable iff there are no eigenvalues in the right complex half-plane, and to all eigenvalues on the imaginary axis correspond Jordan blocks of size 1.
b) asymptotically stable iff all eigenvalues are in the left complex half-plane.



Exercises

1. Draw all possible phase portraits of linear ODEs in $\mathbb{R}^2$.
2. Prove that $\det(e^A) = e^{\operatorname{tr} A}$.
3. May linear operators A and B not commute (i.e. $AB \neq BA$) if
$$e^A = e^B = e^{A+B} = E \quad ?$$
4. Prove that there is no "blow-up" for a linear non-autonomous ODE.

LECTURE 3

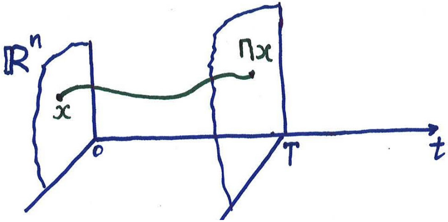
LINEAR DYNAMICAL SYSTEMS

Linear periodic-coefficient ODEs

Consider an ODE

$$\dot{x} = A(t)x,\ x \in \mathbb{R}^n,\ \ A(t+T) = A(t)$$

where $A(t)$ is a linear operator, $A(t)\colon \mathbb{R}^n \to \mathbb{R}^n$, $A(\cdot) \in C^0(\mathbb{R}), T = \mathrm{const} > 0$.
Denote Π the Poincaré return map for the plane $\{t = 0 \bmod T\}$.



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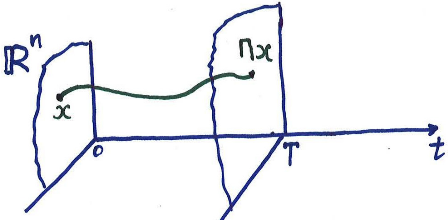

Proposition.

Π is a linear operator, $\Pi\colon \mathbb{R}^n \to \mathbb{R}^n$, Π is non-degenerate and preserves orientation of $\mathbb{R}^n$: $\det \Pi > 0$.

Π is called the *monodromy operator*, its matrix is called the *monodromy matrix*, its eigenvalues are called the *Floquet multilpliers*. The Floquet multilpliers are roots of the *characteristic equation* $\det(\Pi - \rho E) = 0$.

One can complexify both $A(t)$ and Π:
$A(t)\colon \mathbb{C}^n \to \mathbb{C}^n,\ A(t)(x+iy) = A(t)x + iA(t)y,$
$\Pi\colon \mathbb{C}^n \to \mathbb{C}^n,\ \Pi(x+iy) = \Pi x + i\Pi y.$

Let B be a linear operator, $B\colon \mathbb{C}^n \to \mathbb{C}^n$.

Definition

A linear operator $K\colon \mathbb{C}^n \to \mathbb{C}^n$ is called a *logarithm* of B if $B = e^K$.

Theorem (Existence of a logarithm)

Any non-degenerate operator has a logarithm.

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Corollary

Take $\Lambda = \frac{1}{T}\mathrm{Ln}\,\Pi$. *Then* Π *coincides with the evolution operator for the time* T *of the constant-coefficient linear ODE* $\dot z = \Lambda z$.

Eigenvalues of K are called *Floquet exponents*. The relation between Floquet multipliers ρ_j and Floquet exponents λ_j is $\rho_j = e^{T\lambda_j}$. Real parts of Floquet exponents are *Lyapunov exponents*.

Proof.

Because of the theorem about the Jordan normal form it is enough to consider the case when original operator B is a Jordan block. Let ρ be the eigenvalue of this block, $\rho \neq 0$ because of non-degeneracy of B. Then $B = \rho(E + \frac{1}{\rho}I)$, where I is the Jordan block with the eigenvalue 0. Take

$$K = (\mathrm{Ln}\,\rho)E + Y,\;\; Y = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m}(\frac{1}{\rho}I)^m$$

where $\mathrm{Ln}\,\rho = \ln|\rho| + i\mathrm{Arg}\,\rho$. The series for Y actually contains only a finite number of terms because $I^n = 0$. For $z \in \mathbb{C}$ we have $e^{\ln(1+z)} = 1 + z$ and

$$\ln(1+z) = z - \tfrac{1}{2}z^2 + \frac{1}{3}z^3 + \dots,\;\; e^y = 1 + \tfrac{1}{2}y^2 + \frac{1}{3!}y^3 + \dots$$

provided that the series for the logarithm converges. Thus, plugging the series for $y = \ln(1+z)$ to the series for e^y after rearranging of terms gives $1 + z$. Thus,

$$e^K = e^{(\mathrm{Ln}\,\rho)E+Y} = e^{(\mathrm{Ln}\,\rho)E}e^Y = \rho(E + \frac{1}{\rho}I) = B$$

(we use that if $HL = LH$, then $e^{H+L} = e^H e^L$, and that all the series here are absolutely convergent).

Theorem (Existence of real logarithm)

Any non-degenerate real operator which does not have real negative eigenvalues has a real logarithm.

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Any non-degenerate real operator which does not have real negative eigenvalues has a real logarithm.

Proof.

Let Π be our real operator. There is a decomposition of $\mathbb{C}^n$ into a direct sum of invariant subspaces such that in each of subspaces Π acts as a Jordan block:

$$\mathbb{C}^n = V_1 \oplus \ldots \oplus V_r \oplus (V_{r+1} \oplus \bar{V}_{r+1}) \oplus \ldots \oplus (V_k \oplus \bar{V}_k)$$

In this decompositions subspaces $V_1, \ldots, V_r$ correspond to real positive eigenvalues of Π and have real Jordan bases.
Subspaces $V_{r+1}, \ldots, V_k$ correspond to complex eigenvalues, Jordan bases in V_j and $\bar{V}_j$ are chosen to be complex conjugated.
In $\mathrm{Re} V_j$, $j = 1, \ldots, r$ previous formulas allow to define the real logarithm.
In V_j, $\bar{V}_j$, $j = r+1, \ldots, k$ previous formulas allow to define logarithms K and $\bar{K}$ respectively such that if $w \in V_j$ and $\bar{w} \in \bar{V}_j$, then $\overline{Kw} = \bar{K}\bar{w}$. Then we define real logarithm $\mathcal{K}$ acting on $(\mathrm{Re}\, V_j \oplus \mathrm{Im}\, V_j)$ by formulas
$\mathcal{K}\,\mathrm{Re} w = \mathcal{K}\frac{1}{2}(w+\bar{w}) = \frac{1}{2}(Kw+\bar{K}\bar{w})$, $\mathcal{K}\,\mathrm{Im} w = \mathcal{K}\frac{1}{2}\frac{1}{i}(w-\bar{w}) = \frac{1}{2}\frac{1}{i}(Kw-\bar{K}\bar{w})$.
Thus, the real logarithm is defined on
$\mathbb{R}^n = \mathrm{Re} V_1 \oplus \ldots \oplus \mathrm{Re} V_r \oplus (\mathrm{Re}\, V_{r+1} \oplus \mathrm{Im}\, \bar{V}_{r+1}) \oplus \ldots \oplus (\mathrm{Re}\, V_k \oplus \mathrm{Im}\, V_k)$.

Corollary

Square of any non-degenerate real operator has real logarithm.

Corollary

Take $\tilde{\Lambda}$ such that $T\tilde{\Lambda}$ is a real logarithm of Π^2. Then Π^2 coincides with the evolution operator for the time $2T$ of the constant-coefficient linear ODE $\dot{z} = \tilde{\Lambda} z$.

Fix a basis in $\mathbb{R}^n$. It serves also as the basis in $\mathbb{C}^n$ which is complexification of $\mathbb{R}^n$. Identify linear operators with their matrices in this basis. So, now Π is the matrix of monodromy operator, E is the unit matrix.

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Let $X(t)$ be the Cauchy fundamental matrix for our T-periodic equation (i.e. the columns of $X(t)$ are n linearly independent solutions to this equation, $X(0) = E$).

Lemma

a) $X(T) = \Pi$, *b)* $X(t + T) = X(t)\Pi$

Proof.

a) $x(t) = X(t)x(0) \Rightarrow x(T) = X(T)x(0)$. b) $X(t + T) = X(t)R$ for a certain $R = \text{const}$. Plugging $t = 0$ gives $R = X(T) = \Pi$ □

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Theorem (The Floquet theorem)

$X(t) = \Phi(t)e^{t\Lambda}$, *where* $\Phi(\cdot) \in C^1$, $\Phi(t+T) = \Phi(t)$, $\Lambda = \frac{1}{T}\mathrm{Ln}\,\Pi$

Proof.

Take $\Phi(t) = X(t)e^{-t\Lambda}$. Then

$\Phi(t+T) = X(t+T)e^{-(t+T)\Lambda} = X(t)\Pi e^{-T\Lambda}e^{-t\Lambda} = X(t)e^{-t\Lambda} = \Phi(t)$ □

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Remark

The Floquet theorem plays a fundamental role in the solid state physics under the name *the Bloch theorem*.

Theorem (The Lyapunov theorem)

Any linear T-periodic ODE is reducible to a linear constant-coefficient ODE by means of a smooth linear T-periodic transformation of variables.

Proof.

Let matrices $\Phi(t)$ and Λ be as in the Floquet theorem. The required transformation of variables $x \mapsto y$ is given by the formula $x = \Phi(t)y$. The equation in the new variables has the form $\dot{y} = \Lambda y$. □

Remark

Any linear real T-periodic ODE is reducible to a linear real constant-coefficient ODE by means of a smooth linear real $2T$-periodic transformation of variables (because Π^2 has a real logarithm).

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Remark

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Remark

If a T-periodic ODE depends smoothly on a parameter, then the transformation of variables in the Lyapunov theorem can also be chosen to be smooth in this parameter. (Note that this assertion does not follow from the presented proof of the Lyapunov theorem. This property is needed for analysis of bifurcations.)

Linear periodic-coefficient ODEs, stability

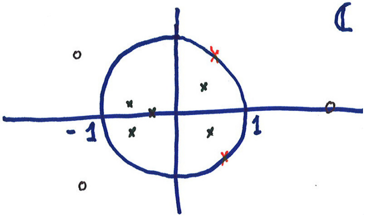
Theorem

A linear periodic-coefficient ODE is
a)stable iff there are no Floquet multipliers outside the unit circle in the complex plane, and to all multipliers on the unit circle correspond Jordan blocks of size 1.
b) asymptotically stable iff all Floquet multipliers are within the unit circle.

Proof.

The Floquet-Lyapunov theory reduces problem of stability for a linear periodic-coefficient ODE to the already solved problem of stability for a linear constant-coefficient ODE. One should note relation $\rho_j = e^{T\lambda_j}$ between Floquet multipliers and Floquet exponents and the fact that the Jordan block structure is the same for operators Λ and $e^{T\Lambda}$.

□



Linear maps

Consider a map

$$x \mapsto \Pi x, \; x \in \mathbb{R}^n,$$

where Π is a non-degenerate linear operator, $\Pi\colon \mathbb{R}^n \to \mathbb{R}^n$, $\det \Pi \neq 0$.

The eigenvalues of Π are called *the multipliers*.

According to the theorem about existence of a logarithm there exists a linear operator $\Lambda\colon \mathbb{C}^n \to \mathbb{C}^n$ such that $\Pi = e^{\Lambda}$. Then $\Pi^k = e^{k\Lambda}$, $k \in \mathbb{Z}$.
This allows to study iterations of linear maps.

One can use also the representation $\Pi^2 = e^{\tilde{\Lambda}}$, where $\tilde{\Lambda}$ is a real linear operator, $\tilde{\Lambda}\colon \mathbb{R}^n \to \mathbb{R}^n$.

Exercises

1. Prove that square of any non-degenerate linear real operator has a real logarithm.

2. Give an example of a real linear ODE with T-periodic coefficients which cannot be transformed into constant-coefficient ODE by any real T-periodic linear transformation of variables.

3. Find $\mathrm{Ln}\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$.

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LECTURE 4

LINEAR DYNAMICAL SYSTEMS

Definition

For a function f, $f : [a, +\infty) \to \mathbb{R}^n$, $a = \mathrm{const}$, the *characteristic Lyapunov exponent* is

$$\chi[f] = \limsup_{t \to +\infty} \frac{\ln |f(t)|}{t}$$

This is either a number or one of the symbols $+\infty, -\infty$.

Example

$\chi[e^{\alpha t}] = \alpha,\ \chi[t^{\beta} e^{\alpha t}] = \alpha,\ \chi[\sin(\gamma t) e^{\alpha t}] = \alpha,\ \chi[e^{t^2}] = +\infty\ (\gamma \neq 0)$

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Proposition.

If functions $f_1, f_2, \ldots, f_n$ have different finite characteristic Lyapunov exponents, then these functions are linearly independent.

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Proposition.

If functions $f_1, f_2, \ldots, f_n$ have different finite characteristic Lyapunov exponents, then these functions are linearly independent.

Definition

The set of the characteristic Lyapunov exponents of all solutions of an ODE is called *the Lyapunov spectrum* of this ODE

Example

Consider equation $\dot{x} = (x/t) \ln x,\ t > 0,\ x > 0$. Its general solution is $x = e^{ct}$ with an arbitrary constant c. So, the Lyapunov spectrum is $\mathbb{R}$.

Consider a linear non-autonomous ODE

$$\dot{x} = A(t)x,\ x \in \mathbb{R}^n$$

where $A(t)$ is a linear operator, $A(t)\colon \mathbb{R}^n \to \mathbb{R}^n$, $A(\cdot) \in C^0(\mathbb{R})$.
Recall that $\|A\| = \sup\limits_{x \neq 0} \|Ax\|/\|x\|$.

Theorem (Lyapunov)

If $\|A(\cdot)\|$ is bounded on $[0, +\infty)$, then each nontrivial solution has a finite characteristic Lyapunov exponent.

Corollary

Lyapunov spectrum of a linear non-autonomous ODE in $\mathbb{R}^n$ with a bounded matrix of coefficients contains no more than n elements.

Exercises

1. Prove that the equation

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 2/t^2 & 0 \end{pmatrix} x$$

can not be transformed into a constant-coefficient linear equation by means of transformation of variables of the form $y = L(t)x$, where $L(\cdot) \in C^1(\mathbb{R}), |L| < \mathrm{const}, |\dot{L}| < \mathrm{const}, |L^{-1}| < \mathrm{const}.$

2. Let $X(t)$ be a fundamental matrix for the equation $\dot{x} = A(t)x$. Prove the Liouville - Ostrogradski formula:

$$\det(X(t)) = \det(X(0))e^{\int_0^t \mathrm{tr}\, A(\tau) d\tau}$$

.

Linear Hamiltonian systems

A Hamiltonian system with a Hamilton function H is ODE system of the form

$$\dot{p} = -\left(\frac{\partial H}{\partial q}\right)^T, \ \dot{q} = \left(\frac{\partial H}{\partial p}\right)^T$$

Here $p \in \mathbb{R}^n$, $q \in \mathbb{R}^n$, p and q are considered as vector-columns, H is a function of p, q, t, the superscript "T" denotes the matrix transposition. Components of p are called "impulses", components of q are called "coordinates".

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$$\dot{x} = J\left(\frac{\partial H}{\partial x}\right)^T, \text{ where } J = \begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}$$

Here E_n is the $n \times n$ unit matrix. The matrix J is called *the symplectic unity*. $J^2 = -E_{2n}$. If H is a quadratic form, $H = \frac{1}{2}x^T A(t)x$, where $A(t)$ is a symmetric $2n \times 2n$ matrix, $A(\cdot) \in C^0$, then we get a *linear Hamiltonian system*

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Theorem (The Liouville theorem)

Shift along solutions of a Hamiltonian system preserves the standard phase volume in $\mathbb{R}^{2n}$.

Corollary

A linear Hamiltonian system can not be asymptotically stable.

Linear Hamiltonian systems, skew-scalar product

Definition

The skew-symmetric bilinear form $[\eta, \xi] = \xi^T J \eta$ in $\mathbb{R}^{2n}$ is called *the skew-scalar product* or *the standard linear symplectic structure*. (Note: $[\eta, \xi] = (J\eta, \xi)$.)

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Theorem

A shift along solutions of a linear Hamiltonian system preserves the skew-scalar product: if $x(t)$ and $y(t)$ are solutions of the system, then $[x(t), y(t)] \equiv \mathrm{const}$.

Proof.

$\frac{d}{dt}[y(t), x(t)] = \frac{d}{dt}x(t)^T Jy(t) = (JAx)^T Jy + x^T JJAy = x^T A^T J^T Jy - x^T Ay = x^T Ay - x^T Ay = 0$ □

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Corollary

Let $X(t)$ be the Cauchy fundamental matrix for our Hamiltonian system (i.e. the columns of $X(t)$ are n linearly independent solutions to this system, $X(0) = E_{2n}$). Then $X^T(t)JX(t) = J$ for all $t \in \mathbb{R}$.

Proof.

Elements of matrix $X^T(t)JX(t)$ are skew-scalar products of solutions, and so this is a constant matrix.

From the condition at $t = 0$ we get $X^T(t)JX(t) = J$. □

Linear symplectic transformations

Definition

A $2n \times 2n$ matrix M which satisfies the relation $M^T JM = J$ is called *a symplectic matrix*. A linear transformation of variables with a symplectic matrix is called *a linear symplectic transformation of variables*.

Corollary

The Cauchy fundamental matrix for linear Hamiltonian system at any moment of time is a symplectic matrix. A shift along solutions of a linear Hamiltonian system is a linear symplectic transformation.

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Theorem

A linear transformations of variables is a symplectic transformations if and only if it preserves the skew-scalar product: $[M\eta, M\xi] = [\eta, \xi]$ *for any* $\xi \in \mathbb{R}^{2n}$, $\eta \in \mathbb{R}^{2n}$; *here* M *is the matrix of the transformation.*

Proof.

$[M\eta, M\xi] = \xi^T M^T JM\eta$ □

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Proof.

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Theorem

Symplectic matrices form a group.

The group of symplectic $2n \times 2n$ matrices is called *the symplectic group of degree* $2n$ and is denoted as $\mathrm{Sp}(2n)$. (The same name is used for the group of linear symplectic transformations of $\mathbb{R}^{2n}$.)

Symplectic transformations in linear Hamiltonian systems

Make in a Hamiltonian system $\dot{x} = JA(t)x$ with a Hamilton function $H = \frac{1}{2}x^T A(t)x$ a symplectic transformation of variables $y = M(t)x$. We have

$$\dot{y} = M\dot{x}+\dot{M}x = MJAx+\dot{M}x = (MJAM^{-1}+\dot{M}M^{-1})y = J(-JMJAM^{-1}-J\dot{M}M^{-1})y$$

Let us show that the obtained equation for y is a Hamiltonian one. Because M is a symplectic matrix, we have

$$M^T JM = J,\ \dot{M}^T JM + M^T J\dot{M} = 0,\ M = -J(M^{-1})^T J,\ \dot{M}^T = M^T J\dot{M}M^{-1}J$$

Thus

$$-JMJAM^{-1} = JJ(M^{-1})^T JJAM^{-1} = (M^{-1})^T AM^{-1},$$

and this is a symmetric matrix. And

$$(J\dot{M}M^{-1})^T = -(M^{-1})^T \dot{M}^T J = -(M^{-1})^T M^T J\dot{M}M^{-1}JJ = J\dot{M}M^{-1}$$

and this also is a symmetric matrix. So, equation for y is a Hamiltonian one.

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and this also is a symmetric matrix. So, equation for y is a Hamiltonian one.

Note, that if $M = \mathrm{const}$, then the Hamilton function for the new system

$$\mathcal{H} = \frac{1}{2}y^T (M^{-1})^T A(t)M^{-1}y$$

is just the old Hamilton function expressed through the new variables.

Constant-coefficient linear Hamiltonian system

Consider an autonomous linear Hamiltonian system;

$$\dot{x} = JAx$$

where A is a constant symmetric $2n \times 2n$ matrix.

Proposition.

The matrix JA is similar to the matrix $(-JA)^T$.

Proof.

$J^{-1}(-JA)^T J = -J^{-1}A^T J^T J = JA$ □

Corollary

If λ is an eigenvalue, then $-\lambda$ is an eigenvalue.
Eigenvalues λ and $-\lambda$ have equal multiplicities and the corresponding Jordan structures are the same.
If $\lambda = 0$ is an eigenvalue, then it necessarily has even multiplicity.

Corollary

Characteristic polynomial $\det(JA - \lambda E_{2n})$ *of a matrix JA is a polynomial in* λ^2.

Constant-coefficient linear Hamiltonian system, location of eigenvalues

Theorem

The eigenvalues of autonomous linear Hamiltonian system are situated on the plane of complex variable symmetrically with respect to the coordinate cross: if λ *is an eigenvalue, then* $-\lambda$, $\bar{\lambda}$, $-\bar{\lambda}$ *are also eigenvalues. The eigenvalues* λ, $-\lambda$, $\bar{\lambda}$, $-\bar{\lambda}$ *have equal multiplicities and the corresponding Jordan structures are the same.*

So, eigenvalues may be of four types: real pairs $(a, -a)$, purely imaginary pairs $(ib, -ib)$, quadruplets $(\pm a \pm ib)$ and zero eigenvalues.

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Corollary

If there is a purely imaginary simple eigenvalue, then it remains on the imaginary axis under a small perturbation of the Hamiltonian. Similarly, a real simple eigenvalue remains real under a small perturbation of the Hamiltonian.

So, the system may loss stability under a small perturbation of the Hamiltonian only if there is $1:1$ resonance.

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LECTURE 5

LINEAR HAMILTONIAN SYSTEMS

List of formulas

A Hamiltonian system with a Hamilton function H is an ODE system of the form

$$\dot{p} = -\left(\frac{\partial H}{\partial q}\right)^T, \ \dot{q} = \left(\frac{\partial H}{\partial p}\right)^T$$

Here $p \in \mathbb{R}^n$, $q \in \mathbb{R}^n$, p and q are considered as vector-columns, H is a function of p, q, t, the superscript "T" denotes the matrix transposition. Components of p are called "impulses", components of q are called "coordinates".

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Let x be the vector-column combined of p and q. Then

$$\dot{x} = J\left(\frac{\partial H}{\partial x}\right)^T, \text{ where } J = \begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}$$

The matrix J is called *the symplectic unity*. $J^2 = -E_{2n}$.

If H is a quadratic form, $H = \frac{1}{2}x^T A(t)x$, where $A(t)$ is a symmetric $2n \times 2n$ matrix, $A(\cdot) \in C^0$, then we get a *linear Hamiltonian system*

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List of formulas, continued

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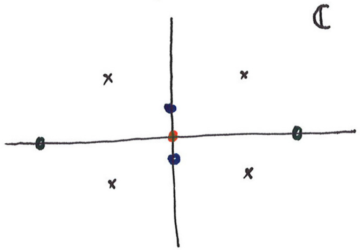

Normal form of quadratic Hamiltonian in the case of pairwise different eigen-frequencies

Let matrix JA has purely imaginary pairwise different eigenvalues $\pm i\omega_1, \pm i\omega_2, \ldots, \pm i\omega_n$. Let ξ_k, $\bar{\xi}_k$ be eigenvectors of JA corresponding to eigenvalues $\pm i\omega_k$.

Theorem

By a certain linear symplectic transformation of variables the Hamilton function $H = \frac{1}{2}x^T Ax$ *can be transformed to the form*

$$H = \frac{1}{2}\Omega_1(p_1^2 + q_1^2) + \frac{1}{2}\Omega_2(p_2^2 + q_2^2) + \ldots + \frac{1}{2}\Omega_n(p_n^2 + q_n^2)$$

where p_k *are "impulses", and* q_k *are "coordinates",* $\Omega_k = \pm\omega_k$.

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Lemma

Let η_1, η_2 be eigenvectors of a matrix JA, and λ_1, λ_2 be the corresponding eigenvalues. If $\lambda_1 \neq -\lambda_2$, then η_1 and η_2 are scew-orthogonal: $[\eta_2, \eta_1] = 0$.

Proof.

$JA\eta_k = \lambda_k\eta_k$. Thus, $A\eta_k = -\lambda_k J\eta_j$, and $\eta_1^T A\eta_2 = -\lambda_2\eta_1^T J\eta_2 = \lambda_1\eta_1^T J\eta_2$ □

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Lemma

$[\xi_k, \bar{\xi}_k]$ is purely imaginary number, $[\xi_k, \bar{\xi}_k] \neq 0$.

Proof.

The complex conjugation change sign of $[\xi_k, \bar{\xi}_k]$.

Proof of the theorem.

Without loss of generality we may assume that $[\xi_k, \bar{\xi}_k]$ is equal to either i or $-i$. Introduce $a_k = (\xi_k + \bar{\xi}_k)/\sqrt{2}$, $b_k = (\xi_k - \bar{\xi}_k)/(\sqrt{2}i)$.

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Choose as the new basis in $\mathbb{R}^{2n}$ vectors $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$ (in this order). The matrix of transformation of vector's coordinates for this change of basis is a symplectic one.

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The form of the Hamiltonian shows that the phase space is a direct product of of invariant two-dimensional mutually skew-orthogonal planes. Dynamics in kth plane is decribed by the Hamilton function $H_k = \frac{1}{2}\Omega_k(p_k^2 + q_k^2)$; it is called *a partial Hamilton function*, or just *a partial Hamiltonian*. This is the Hamilton function of a linear oscillator with the frequency Ω_k. In the plane q_k, p_k phase curves are circles. Rotation on this circles is clockwise, if $\Omega_k > 0$, and counterclockwise, if $\Omega_k < 0$. Values $|\Omega_k|$ are called *eigen-frequencies* of the system. Phase space $\mathbb{R}^{2n}$ is foliated (with singularities) by n-dimensional invariant tori.

Theorem (Williamson)

There exists a real symplectic linear change of variables reducing the Hamiltonian to a sum of partial Hamiltonians (functions of disjoint subsets of conjugate variables), and the matrix of the system, correspondingly, to a block-diagonal form. Each partial Hamiltonian corresponds either to a real pair, or to an imaginary pair, or to a quadruplet of eigenvalues, or to a zero eigenvalue. The partial Hamiltonians are determined, up to a sign, by the Jordan blocks of the operator JA.

The list of partial Hamiltonians is contained in the paper by J.Williamson (1936).

When parameters of a stable constant-coefficient linear Hamiltonian system are changing, the eigenvalues move on the imaginary axis. When eigenvalues collide they may leave the imaginary axis. However, collision of not each pair of eigenvalues is dangerous, some eigenvalues necessarily pass through each other without leaving the imaginary axis.

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While all eigenvalues are simple the Hamilton function can be reduced to the normal form

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We say that a partial Hamilton function $H_k = \frac{1}{2}\Omega_k(p_k^2 + q_k^2)$ is positive (negative), if Ω_k is positive (respectively, negative).

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Theorem

If imaginary eigenvalues corresponding to partial Hamilton functions having the same sign collide not at the point 0, the system does not loss its stability. In particular, when these eigenvalues collide, the matrix of the Hamiltonian system does not have Jordan blocks of sizes bigger than 1.

For *a natural Hamiltonian system* its Hamilton function H is the sum of a kinetic energy $T = T(p, q) = \frac{1}{2} p^T K^{-1}(q) p$ and a potential energy $V(q)$:

$$H = T + V.$$

Here $K(q)$ is a positive definite matrix.

On linear natural Hamiltonian systems

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Here $K(q)$ is a positive definite matrix. For a linear natural Hamiltonian system $K = \mathrm{const}$ and V is a quadratic form: $V = \frac{1}{2}q^T Dq$.

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Corollary

For linear natural Hamiltonian system collision of non-zero eigenvalues on the imaginary axis does not lead to a stability loss. Collision of eigenvalues at 0 leads to the stability loss.

Proof of the theorem.

There exists a matrix C which simultaneously transforms the quadratic form $y^T Ky$ to sum of squares, and the quadratic form $y^T Dy$ to a diagonal form, i.e.

$$C^T KC = E_n,\ \ C^T DC = \mathrm{diag}(\mu_1, \mu_2, \dots, \mu_n)$$

The transformation of variables $p, q \mapsto P, Q,\ \ p = (C^{-1})^T P,\ q = CQ$ is a symplectic one. Indeed, if $\tilde{p} = (C^{-1})^T \tilde{P},\ \tilde{q} = C\tilde{Q}$, then

$$\tilde{P}^T Q - \tilde{Q}^T P \equiv \tilde{p}^T q - \tilde{q}^T p$$

The Hamilton function in the new variables is

$$H = \tfrac{1}{2}(P_1^2 + P_2^2 + \dots + P_n^2) + \tfrac{1}{2}(\mu_1 Q_1^2 + \mu_2 Q_2^2 + \dots + \mu_n Q_n^2)$$

It is the sum of the partial Hamilton functions of the form $H_k = \frac{1}{2}(P_k^2 + \mu_k Q_k^2)$. The equation of the motion for a Hamilton function H_k has the form

$$\ddot{Q}_k + \mu_k Q_k = 0$$

Eigenvalues are $\pm i\sqrt{\mu}_k$, if $\mu_k > 0$, and $\pm\sqrt{-\mu_k}$, if $\mu_k < 0$. Only for $\mu_k = 0$ the solution contains multiplier t, and so only to eigenvalue 0 there correspond the Jordan block of the size 2.

Exercises

1. Consider an equation $\dot{x} = v(t, x),\ x \in \mathbb{R}^k$. Prove that if $\mathrm{div} v \equiv 0$, then the shift along trajectories of this equation preserves the standard phase volume in $\mathbb{R}^k$. (Hint: $\mathrm{div} v = \mathrm{tr}(\partial v / \partial x)$.)

2. Give a geometric interpretation for skew-scalar product in the case of one degree of freedom (n=1) and in the case of several degrees of freedom.

3. Obtain formulas for partial Hamilton functions corresponding to the pair of a simple real eigenvalues ($\pm a$), to the quadruplet of a simple complex eigenvalues ($\pm a \pm ib$) and to the eigenvalue 0 of multiplicity 2 ($a \neq 0$, $b \neq 0$).

Periodic-coefficient linear Hamiltonian system.

Consider a periodic linear Hamiltonian system;

$$\dot{x} = JA(t)x$$

where $A(t)$ is a symmetric $2n \times 2n$ matrix, $A(t+T) = A(t)$, $A(\cdot) \in C^0$.

Let $X(t)$ be the Cauchy fundamental matrix for this system (i.e. the columns of $X(t)$ are n linearly independent solutions to this system, $X(0) = E_{2n}$).

We know that $X^T(t)JX(t) \equiv J$. This implies that $\det X(t) \equiv 1$.

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Let Π be the monodromy matrix for this system. We know that $\Pi = X(T)$.
Thus $\Pi^T J\Pi = J$ (so, Π is a symplectic matrix) and $\det \Pi = 1$

Recall the definition: Multipliers a eigenvalues of the monodromy matrix.

Proposition.

The matrix Π is similar to the matrix $\left(\Pi^{-1}\right)^T$.

Proof.

$\Pi^T J \Pi = J$ implies that $\left(\Pi^{-1}\right)^T = J \Pi J^{-1}$ □

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Corollary

If ρ is a multiplier , then $1/\rho$ is a multiplier.
Multipliers ρ and $1/\rho$ have equal multiplicities and the corresponding Jordan structures in Π are the same.
If $\rho = -1$ is a multiplier, then it necessarily has even multiplicity.
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Periodic-coefficient linear Hamiltonian system, properties of multipliers

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Periodic-coefficient linear Hamiltonian system, location of multipliers

Theorem

Multipliers of a periodic-coefficient linear Hamiltonian system are situated on the plane of complex variable symmetrically with respect to the unit circle (in the sense of the inversion) and with respect to the real axis: if ρ is a multiplier, then $1/\rho$, $\bar{\rho}$, $1/\bar{\rho}$ are also multipliers. The multipliers ρ, $1/\rho$, $\bar{\rho}$, $1/\bar{\rho}$ have equal multiplicities and the corresponding Jordan structures are the same.

So, multipliers may be of five types: real pairs $(a, 1/a)$, pairs on the unit circle $e^{i\phi}, e^{-i\phi}$, quadruplets $(a^{\pm 1}e^{\pm i\phi})$, -1, and 1.

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So, the system may loss stability under a small perturbation of the Hamiltonian only if there is a resonance.

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When parameters of a stable periodic-coefficient linear Hamiltonian system are changing, the multipliers move on the unit circle. When multipliers collide they may leave the unit circle; this result in loss of stability. However, collision of not each pair of multipliers is dangerous, some multipliers necessarily pass through each other without leaving the unit circle (M.G.Krein's theory).

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LECTURE 6

LINEAR HAMILTONIAN SYSTEMS

A Hamiltonian system with a Hamilton function H is an ODE system of the form

$$\dot{p} = -\left(\frac{\partial H}{\partial q}\right)^T, \ \dot{q} = \left(\frac{\partial H}{\partial p}\right)^T$$

Here $p \in \mathbb{R}^n$, $q \in \mathbb{R}^n$, p and q are considered as vector-columns, H is a function of p, q, t, the superscript "T" denotes the matrix transposition. Components of p are called "impulses", components of q are called "coordinates".

List of formulas

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Let x be the vector-column combined of p and q. Then

$$\dot{x} = J\left(\frac{\partial H}{\partial x}\right)^T, \text{ where } J = \begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}$$

The matrix J is called *the symplectic unity*. $J^2 = -E_{2n}$.

If H is a quadratic form, $H = \frac{1}{2}x^T A(t)x$, where $A(t)$ is a symmetric $2n \times 2n$ matrix, $A(\cdot) \in C^0$, then we get a *linear Hamiltonian system*

$$\dot{x} = JA(t)x$$

The skew-symmetric bilinear form $[\eta, \xi] = \xi^T J\eta$ in $\mathbb{R}^{2n}$ is called *the skew-scalar product* or *the standard linear symplectic structure*.

A $2n \times 2n$ matrix M which satisfies the relation $M^T JM = J$ is called *a symplectic matrix*. A linear transformation of variables with a symplectic matrix is called *a linear symplectic transformation of variables*.

A linear transformations of variables is a symplectic transformations if and only if it preserves the skew-scalar product: $[M\eta, M\xi] = [\eta, \xi]$ for any $\xi \in \mathbb{R}^{2n}$, $\eta \in \mathbb{R}^{2n}$; here M is the matrix of the transformation.

The Cauchy fundamental matrix $X(t)$ for a linear Hamiltonian system (i.e. the matrix $X(t)$ such that its columns are n linearly independent solutions to this system, $X(0) = E_{2n}$) is a symplectic matrix , i. e. $X^T(t)JX(t) = J$ for all $t \in \mathbb{R}$.

For any linear T periodic ODE the matrix of the monodromy map is equal to the value of Cauchy fundamental matrix $X(t)$ at the moment of time T: $\Pi = X(T)$.

Consider a periodic linear Hamiltonian system;

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where $A(t)$ is a symmetric $2n \times 2n$ matrix, $A(t+T) = A(t)$, $A(\cdot) \in C^0$.

Let $X(t)$ be the Cauchy fundamental matrix for this system (i.e. the columns of $X(t)$ are n linearly independent solutions to this system, $X(0) = E_{2n}$).

We know that $X^T(t)JX(t) \equiv J$. This implies that $\det X(t) \equiv 1$.

Periodic-coefficient linear Hamiltonian system.

Consider a periodic linear Hamiltonian system;

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Multipliers of a periodic-coefficient linear Hamiltonian system are situated on the plane of complex variable symmetrically with respect to the unit circle (in the sense of the inversion) and with respect to the real axis: if ρ is a multiplier, then $1/\rho$, $\bar{\rho}$, $1/\bar{\rho}$ are also multipliers. The multipliers ρ, $1/\rho$, $\bar{\rho}$, $1/\bar{\rho}$ have equal multiplicities and the corresponding Jordan structures are the same.

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When parameters of a stable periodic-coefficient linear Hamiltonian system are changing, the multipliers move on the unit circle. When multipliers collide they may leave the unit circle; this result in loss of stability. However, collision of not each pair of multipliers is dangerous, some multipliers necessarily pass through each other without leaving the unit circle (M.G.Krein's theory).

Periodic-coefficient linear Hamiltonian system, Krein's theory

In the Krein's theory each simple multiplier on the unit circle has a property which is called *the Krein signature*.
The Krein signature may be either *positive* or *negative*.
If two multipliers on the unit circle with the same Krein signature collide, they pass through each other. If two multipliers on the unit circle with different Krein signatures collide, they typically leave the unit circle; the multiplier with the positive Krein signature is moving inside the unit circle, and the multiplier with the negative Krein signature is moving outside the unit circle.

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One can distinguish multipliers with different Krein signatures on the basis of the following test:
Consider perturbed system

$$\dot{x} = J(A(t) + \delta B(t))x$$

$B(t)$ is a symmetric *positive-definite* $2n \times 2n$ matrix, $B(t+T) = B(t)$, $0 < \delta \ll 1$.

Theorem (Krein's test)

For small enough δ, when δ groves, on the unit circle multipliers with the positive signature move counterclockwise, and multipliers with the negative signature move clockwise.

Consider a periodic-coefficient linear Hamiltonian system which is a small time-periodic perturbation of a stable constant-coefficient Hamiltonian system:

$$\dot{x} = J(A + \varepsilon B(t))x$$

where A and $B(t)$ are a symmetric $2n \times 2n$ matrices, $B(t + T) = B(t)$, $B(\cdot) \in C^0$, $0 < \varepsilon \ll 1$.

We assume that the eigenvalues of JA are different purely imaginary numbers $\pm i\omega_1, \pm i\omega_2, \dots, \pm i\omega_n$. This guarantees stability of the unperturbed system. Values $|\omega_1|, |\omega_2|, \dots, |\omega_n|$ are the eigen-frequencies of the unperturbed system.

Parametric resonance

Consider a periodic-coefficient linear Hamiltonian system which is a small time-periodic perturbation of a stable constant-coefficient Hamiltonian system:

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If B were constant, then the system would be stable provided that ε is small enough. It turns out that the system can be made unstable by an arbitrary small periodic perturbation of an appropriate period T. This phenomenon is called *the parametric resonance*. **The necessary condition for the parametric resonance is that the unperturbed system considered as T-periodic system has non-simple multipliers.**

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The multipliers are $e^{\pm i\omega_1 T}, e^{\pm i\omega_2 T}, \dots, e^{\pm i\omega_n T}$. So, in the case of a parametric resonance $e^{\pm i\omega_j T} = e^{\pm i\omega_m T}$ for some j, m. This implies that $\pm\omega_j T = \pm\omega_m T + 2\pi k$ for a certain integer k. For $j = m$ we get $\omega_m = \frac{k}{2}\frac{2\pi}{T}$.

Parametric resonance, application of the Krein theory

Let the normal form of the Hamiltonian function for the Hamiltonian system $\dot{x} = JAx$ be

$$H = \tfrac{1}{2}\Omega_1(p_1^2 + q_1^2) + \tfrac{1}{2}\Omega_2(p_2^2 + q_2^2) + \ldots + \tfrac{1}{2}\Omega_n(p_n^2 + q_n^2)$$

Then eigenvalues of the matrix JA are $\pm i\Omega_1, \pm i\Omega_2, \ldots, \pm i\Omega_n$.
Consider Hamiltonian system whose Hamiltonian in the variables p, q is

$$F = H + \tfrac{1}{2}\delta(p_1^2 + q_1^2 + p_2^2 + q_2^2 + \ldots + p_n^2 + q_n^2)$$

The eigenvalues of the matrix of this Hamiltonian system are

$$\pm i(\Omega_1 + \delta), \pm i(\Omega_2 + \delta), \ldots, \pm i(\Omega_n + \delta).$$

The eigenvalues of the monodromy matrix for the time T are

$$e^{\pm i(\Omega_1+\delta)T}, e^{\pm i(\Omega_2+\delta)T}, \ldots, e^{\pm i(\Omega_n+\delta)T}.$$

According to the Krein test, when δ grows, the multipliers with the positive Krein signature should move counterclockwise, and multipliers with the negative signature should move clockwise. Therefore, here multipliers with the sign "+" in the exponent have positive Krein signature, and those with the sign "-" have negative Krein signature.
Thus, if a relation $e^{i\Omega_j T} = e^{i\Omega_m T}$ for some j, m is satisfied, then a T-periodic perturbation can not destroy stability (there is no parametric resonance). In opposite, if a relation $e^{i\Omega_j T} = e^{-i\Omega_m T}$ for some j, m is satisfied, then there exists a T-periodic perturbation that destroy stability, i.e. leads to a parametric resonance.

Symplectic version of Floquet-Lyapunov theory

Theorem

A linear Hamiltonian system that is T-periodic in time can be reduced to an autonomous form by a linear symplectic transformation of variables. If the system has no negative real multipliers, then the reducing transformation of variables can be chosen to be T-periodic in time, and if the system has negative real multipliers, then $2T$-periodic. If the system depends smoothly on a parameter, then the transformation of variables can also be chosen to be smooth in this parameter.

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Remark

Let a periodic-coefficient linear Hamiltonian system has the form

$$\dot{x} = J(A + \varepsilon B(t, \varepsilon))x$$

with a smooth dependence of B on ε. Then, according to the previous theorem it can be reduced to an autonomous system

$$\dot{y} = J(A_0 + \varepsilon A_1(\varepsilon))y$$

However, it can happens that the reducing transformation is not close to the autonomous one and $A_0 \neq A$. The reason is similar to that for a parametric resonance. It can happens that the matrix A has simple eigenvalues λ_1 and λ_2 such that $\lambda_2 - \lambda_1 = \frac{2\pi i}{T}m$ with an integer m. Then for $\varepsilon = 0$ the monodromy matrix $\Pi = e^{TA}$ has the eigenvalue (the multiplier) $e^{T\lambda_1}$ of multiplicity 2.

Exercises

1. Consider an equation $\ddot{y} + p(t)y = 0,\ y \in \mathbb{R},\ p(t+T) = p(t)$. Let Π be a monodromy matrix for this equation (what is the definition of a monodromy matrix for this equation?). Prove that if $|\mathrm{tr}\,\Pi| < 2$, then this equation is stable, and if $|\mathrm{tr}\,\Pi| > 2$, it is unstable.

2. Let in the previous exercise $p(t+2\pi) = p(t)$, $\varepsilon \ll 1$ and

$$p(t) = \begin{cases} (\omega+\varepsilon)^2, & \text{if } 0 \le t < \pi; \\ (\omega-\varepsilon)^2, & \text{if } \pi \le t < 2\pi. \end{cases}$$

Find domains of stability in the plane ω, ε.

LINEARISATION

Variation equation

Consider an ODE

$$\dot{x} = v(x),\ x \in D \subset \mathbb{R}^n,\ \ v \in C^2(D)$$

Denote $\{g^t\}$ the phase flow associated with this equation. For an arbitrary $x_* \in D$ consider the solution $g^t x_*,\ t \in \mathbb{R}$ to this ODE.
Introduce $\xi = x - g^t x_*$. Then

$$\dot{\xi} = v(g^t x_* + \xi) - v(g^t x_*) = \frac{\partial v(g^t x_*)}{\partial x}\xi + O(|\xi|^2)$$

Denote $A(t) = \frac{\partial v(g^t x_*)}{\partial x}$

Definition

The linear non-autonomous ODE

$$\dot{\xi} = A(t)\xi$$

is called *the variation equation* near the solution $g^t x_*$.

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Definition

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Denote $\varphi(t) = \frac{dg^t x_*}{dt}$.

Proposition.

$\varphi(t)$ is a solution to the variation equation.

Proof.

$\dot{\varphi} = \frac{d}{dt}\left(\frac{dg^t x_*}{dt}\right) = \frac{d}{dt} v(g^t x_*) = \frac{\partial v(g^t x_*)}{\partial x}\frac{dg^t x_*}{dt} = A(t)\varphi$ □

Definition

A point $x_* \in D$ is called *an equilibrium position* for the equation $\dot{x} = v(x)$ if $v(x_*) = 0$, or, equivalently, $g^t x_* = x_* \; \forall t \in \mathbb{R}$.

The variation equation near the solution $x \equiv x_*$ is the linear constant-coefficient ODE

$$\dot{\xi} = A\xi, \; A = \frac{\partial v(x_*)}{\partial x}$$

This equation is called the linearisation of the original ODE near the equilibrium position x_*.

Definition

A solution $g^t x_*,\ t \in \mathbb{R}$ to the equation $\dot{x} = v(x)$ is called *a T-periodic solution*, if $g^t x_*$ is a periodic function of the time t with the minimal period T. The trajectory of this solution is the closed curve which is called a T-periodic trajectory (or an orbit, or a cycle).

The variation equation near such periodic solution is the linear periodic-coefficient ODE

$$\dot{\xi} = A\xi,\ A = \frac{\partial v(g^t x_*)}{\partial x}$$

This equation is called the linearisation of the original ODE near the periodic solution $g^t x_*$, $t \in \mathbb{R}$.

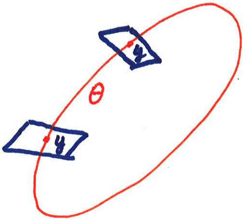
Note that $\varphi(t) = \frac{dg^t x_*}{dt}$ is a T-periodic solution to the variation equation.

Normal coordinates in a neighborhood of periodic trajectory

Theorem

In a small neighborhood of any periodic trajectory there exists a transformation of variables $x \mapsto (y, \theta)$, $y \in \mathbb{R}^{n-1}$, $\theta \in \mathbb{S}^1$ *such that the system takes the form*

$$\begin{aligned} \dot{y} &= u(y, \theta), \ u(0, \theta) \equiv 0 \\ \dot{\theta} &= \omega + w(y, \theta), \ \omega = \text{const} > 0, \ w(0, \theta) \equiv 0 \end{aligned}$$



The variables y in such representation are called *normal variables*. The periodic trajectory has the equation $y = 0$ and is parametrized by the angle θ.

Variation equation near periodic trajectory in normal coordinates

Using normal coordinates in a neighborhood of a periodic trajectory with θ as the new time leads to the equation

$$\frac{dy}{d\theta} = \frac{1}{\omega} u(y, \theta) + O(|y|^2)$$

Denote $B(\theta) = \frac{1}{\omega} \frac{\partial u(0,\theta)}{\partial y}$.

Definition

The linear 2π-periodic ODE

$$\frac{dy}{d\theta} = B(\theta) y$$

is called the linearisation, or the variation equation, of the original ODE near the periodic trajectory in the normal coordinates.

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According to Floquet-Lyapunov theory this ODE can be transformed into an autonomous ODE by a linear 2π- or 4π-periodic in θ transformation of variables.

LECTURE 7

LINEARISATION

Variation equation

Consider an ODE

$$\dot{x} = v(x),\ x \in D \subset \mathbb{R}^n,\ \ v \in C^2(D)$$

Denote $\{g^t\}$ the phase flow associated with this equation. For an arbitrary $x_* \in D$ consider the solution $g^t x_*,\ t \in \mathbb{R}$ to this ODE.
Introduce $\xi = x - g^t x_*$. Then

$$\dot{\xi} = v(g^t x_* + \xi) - v(g^t x_*) = \frac{\partial v(g^t x_*)}{\partial x}\xi + O(|\xi|^2)$$

Denote $A(t) = \frac{\partial v(g^t x_*)}{\partial x}$

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The linear non-autonomous ODE

$$\dot{\xi} = A(t)\xi$$

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Denote $\varphi(t) = \frac{dg^t x_*}{dt}$.

Proposition.

$\varphi(t)$ is a solution to the variation equation.

Proof.

$\dot{\varphi} = \frac{d}{dt}\left(\frac{dg^t x_*}{dt}\right) = \frac{d}{dt}v(g^t x_*) = \frac{\partial v(g^t x_*)}{\partial x}\frac{dg^t x_*}{dt} = A(t)\varphi$ $\square$

Linearisation of ODE near equilibrium position

Definition

A point $x_* \in D$ is called *an equilibrium position*, or just *an equilibrium*, for the equation $\dot{x} = v(x)$ if $v(x_*) = 0$, or, equivalently, $g^t x_* = x_* \, \forall t \in \mathbb{R}$.

The variation equation near the solution $x \equiv x_*$ is the linear constant-coefficient ODE

$$\dot{\xi} = A\xi, \;\; A = \frac{\partial v(x_*)}{\partial x}$$

This equation is called the linearisation of the original ODE near the equilibrium x_*.

Eigenvalues of the linear operator A are called eigenvalues of the equilibrium x_*.

Definition

A solution $g^t x_*,\ t \in \mathbb{R}$ to the equation $\dot{x} = v(x)$ is called *a T-periodic solution*, if $g^t x_*$ is a periodic function of the time t with the minimal period T. The trajectory of this solution is the closed curve which is called a T-periodic trajectory (or an orbit, or a cycle).

The variation equation near such periodic solution is the linear periodic-coefficient ODE

$$\dot{\xi} = A(t)\xi,\ A(t) = \frac{\partial v(g^t x_*)}{\partial x}$$

This equation is called the linearisation of the original ODE near the periodic solution $g^t x_*,\ t \in \mathbb{R}$.

Mutlipliers of this linear periodic-coefficient ODE are called multipliers of the periodic solution $g^t x_*,\ t \in \mathbb{R}$.

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Thus, periodic solution always have 1 as a multiplier.

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In a small neighborhood of any periodic trajectory there exists a transformation of variables $x \mapsto (y, \theta)$, $y \in \mathbb{R}^{n-1}$, $\theta \in \mathbb{S}^1$ such that the system takes the form

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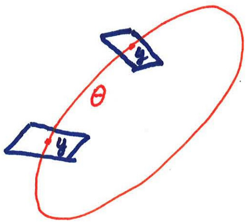

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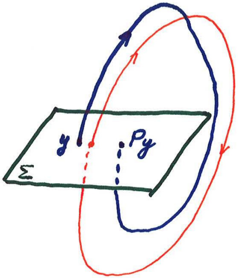
According to Floquet-Lyapunov theory this ODE can be transformed into an autonomous ODE by a linear 2π- or 4π-periodic in θ transformation of variables.

Multipliers of the variation equation near the periodic trajectory in the normal coordinates are called multipliers of the periodic trajectory. (Note: this multipliers are also multipliers of corresponding periodic solution, which have also 1 as the multiplier.)

Poincaré return map near periodic trajectory

Use normal coordinates in a neighborhood of the periodic trajectory. Take $\theta = 0$ as the surface of the Poincaré section. Consider the Poincaré first return map. (In this case it is called also *the monodromy map*.)



The Poincaré map has the form

$$y \mapsto Ky + O(|y|^2)$$

where K is a linear operator, $K : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$.

The operator K is the shift operator during the "time" $\theta = 2\pi$ for the variation equation in the normal coordinates. It is non-degenerate and homotopic to the identity. So, it preserves the orientation of $\mathbb{R}^{n-1}$: $\det K > 0$.

Linearisation of map near fixed point

Consider a map

$$P : D \to D, \ D \subset \mathbb{R}^n, \ P \in C^2(D)$$

This map sends a point x to a point $P(x)$.

Definition

The point $x_* \in D$ is called *a fixed point* of the map P if $P(x_*) = x_*$.

Introduce $\xi = x - x_*$. Then

$$x_* + \xi \mapsto P(x_* + \xi) = x_* + \frac{\partial P(x_*)}{\partial x}\xi + O(|\xi|^2)$$

Denote $A = \frac{\partial P(x_*)}{\partial x}$

Definition

The linear map

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Multipliers of the fixed point of Poincare return map near a periodic trajectory coincide with the defined earlier mutipliers of this periodic trajectory.

Linearisation of Hamiltonian system near equilibrium position

Consider a Hamiltonian system with a Hamilton function H :

$$\dot{p} = -\left(\frac{\partial H}{\partial q}\right)^T, \ \dot{q} = \left(\frac{\partial H}{\partial p}\right)^T$$

Let a point $p = p_*$, $q = q_*$ be an equilibrium position of this system. Then at this point partial derivatives of the Hamilton function vanish, and so this point is a critical point of the Hamilton function. Linearised near the equilibrium position system is again a Hamiltonian system. Its Hamilton function is the quadratic part of the original Hamilton function near the equilibrium position.

Linearisation of Hamiltonian system near periodic trajectory

Suppose that an autonomous Hamiltonian system in $\mathbb{R}^{2n}$ has a periodic trajectory which is not an equilibrium position. **Such trajectories are not isolated but, as a rule, form families**.

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Proposition.

In a neighbourhood of a periodic trajectory there exist new symplectic coordinates $\theta \bmod 2\pi$, $I \in \mathbb{R}^1$ and $z \in \mathbb{R}^{2(n-1)}$, such that $I = 0$ and $z = 0$ on the trajectory under consideration, and going around this trajectory changes θ by 2π; on the trajectory itself $\dot{\theta} = \Omega = \mathrm{const} > 0$. In the new coordinates the Hamiltonian function takes the form $H = \Omega I + \mathcal{H}(z, \theta, I)$, where the expansion of $\mathcal{H}$ in z, I begins with terms of the second order of smallness.

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Perform the *isoenergetic reduction* (also called *Whittaker transformation*) choosing, on each energy level $H = h$ close to the original one, the phase θ for the new time. The Hamiltonian of the problem takes the form $F = F(z, \theta, h)$. For $h = 0$ the origin is an equilibrium position of the system.

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$$F = \frac{1}{2} z^T A(\theta, h) z + G(z, \theta, h)$$

where the expansion of G in z begins with terms of the third order of smallness; the Hamiltonian has period 2π in θ.

Linearisation of Hamiltonian system near periodic trajectory, continued

Symplectic version of the Floquet-Lyapunov theory allows to reduce the problem to the case when the quadratic part of the Hamiltonian does not depend on θ:

$$\tilde{F} = \frac{1}{2} z^T \tilde{A}(h) z + G(z, \theta, h) \,.$$

Exercises

1. Prove that multipliers of a periodic trajectory coincide with multipliers of the fixed point of the corresponding Poincaré return map.

2. Let for isoenergetic reduction procedure some coordinate is introduced as the new time. Proof that the new Hamilton function is the impulse conjugated to this coordinate, expressed through value of energy and remaining phase variables and taken with the sign "-".

LOCAL TOPOLOGICAL EQUIVALENCE TO LINEAR SYSTEM

Definition of topological equivalence

Definition

Two dynamical systems with phase spaces $X_1 \subseteq \mathbb{R}^n$ and $X_2 \subseteq \mathbb{R}^n$ are *topologically equivalent* in the domains $U_1 \subseteq X_1$, $U_2 \subseteq X_2$ if there exists a homeomorphism $\eta \colon U_1 \to U_2$ which maps trajectories, half-trajectories, segments of trajectories of the first system to trajectories, half-trajectories, segments of trajectories of the second system preserving the direction of time.

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For dynamical systems with discrete time the topological equivalency is equivalent to *the topological conjugacy* of evolutionary operators for time 1.

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Definition

Two diffeomorphisms, g_1 and g_2, which are defined in the domains $X_1 \subseteq \mathbb{R}^n$ and $X_2 \subseteq \mathbb{R}^n$ respectively, are *topologically conjugate* in the domains $U_1 \subseteq X_1$, $U_2 \subseteq X_2$ if there exists a homeomorphism $\eta \colon U_1 \to U_2$ such that $\eta(g_1(x)) = g_2(\eta(x))$ for any $x \in U_1$.

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If g_1 and g_2 are topologically conjugate then we have the commutative diagram:

$$
\begin{array}{ccccccc}
\longrightarrow & U_1 & \xrightarrow{g_1} & U_1 & \xrightarrow{g_1} & U_1 & \longrightarrow \\
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Local topological equivalence to linearisation

A question about local topological equivalence to linearisation make sense only for neighborhoods of equilibria without eigenvalues on imaginary axis, for periodic trajectories (fixed points of maps) without multipliers on the unit circle.

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Definition

An equilibrium position of an ODE is called *hyperbolic* if it does not have eigenvalues on the imaginary axis.
A periodic trajectory of an ODE is called *hyperbolic* if it does not have multipliers on the unit circle.
A fixed point of a map is called *hyperbolic* if it does not have multipliers on the unit circle.

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LECTURE 8

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Local topological equivalence to linearisation, Grobman-Hartman theorem

Theorem (The Grobman-Hartman theorem)

Any autonomous ODE with a C^1-smooth right hand side in a neighborhood of a hyperbolic equilibrium position (or a hyperbolic periodic trajectory) is topologically equivalent to its linearisation near this equilibrium position (respectively, near this periodic trajectory).
Any C^1-smooth map in a neighborhood of a hyperbolic fixed point is topologically conjugate to its linearisation near this fixed point.

Definition

A topological type of a hyperbolic equilibrium is a pair (n_s, n_u), were n_s and n_u are the numbers of the eigenvalues of this equilibrium in the left and right complex half-plane respectively (note: "s" is for "stable", "u" is for "unstable").

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Theorem

Linear (hyperbolic) autonomous ODEs in $\mathbb{R}^n$ of the same topological type are topologically equivalent.

Definition

A topological type of a hyperbolic equilibrium is a pair (n_s, n_u), were n_s and n_u are the numbers of the eigenvalues of this equilibrium in the left and right complex half-plane respectively (note: "s" is for "stable", "u" is for "unstable").

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Any autonomous ODE in $\mathbb{R}^n$ with a C^1-smooth right hand side in a neighborhood of a hyperbolic equilibrium position of a topological type (n_s, n_u) is topologically equivalent to the system of linear ODEs

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A hyperbolic equilibrium position of a topological type (n_s, n_u) is called *a topological saddle* if both n_s and n_u are different from 0. It is called a stable (respectively, unstable) *topological node* if $n_u = 0$ (respectively, $n_s = 0$).

Topological classification of maps near hyperbolic fixed points

Definition

A topological type of a hyperbolic fixed point of a map is a quadruple $(n_s, \delta_s, \ n_u, \delta_u)$, were n_s and n_u are the numbers of the multipliers of this fixed point inside and outside of the unit circle respectively, δ_s and δ_u are *signs* of products of these multipliers.

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$$\xi \mapsto A_s\xi,\; \xi \in \mathbb{R}^{n_s},\; \eta \mapsto A_u\eta,\; \eta \in \mathbb{R}^{n_u},$$

where A_s is the diagonal matrix with all diagonal elements equal to $1/2$ but the last one which is equal to $\delta_s/2$, and A_u is the diagonal matrix with all diagonal elements equal to 2 but the last one which is equal to $2\delta_u$.

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For the linear maps above the plane $\{\eta = 0\}$ is called the stable plane, and the plane $\{\xi = 0\}$ is called the unstable plane. The maps with $\delta_s = -1$, (respectively, with $\delta_u = -1$) change the orientation of the stable (respectively, unstable) plane.

The maps with different topological types are topologically not equivalent.

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Because any monodromy map in $\mathbb{R}^n$ preserves the orientation of $\mathbb{R}^n$, for such map necessarily $\delta_s \delta_u = 1$. So, for given $(n_s,\, n_u)$ there are two topologically non-equivalent classes of ODEs near hyperbolic periodic trajectories.

Exercises

1. Provide explicit formulas for homeomorphism of phase portraits of systems

$$\dot{x}_1 = -x_1, \ \dot{x}_2 = -x_2/3$$

and

$$\dot{x}_1 = -x_1 + x_2, \ \dot{x}_2 = -x_1 - x_2$$

2. Prove that eigenvalues of equilibria and multipliers of periodic trajectories are invariant under smooth transformations of variables.

LOCAL INVARIANT MANIFOLDS

Definition

A manifold is called *an invariant manifold* for a dynamical system if this manifold together with each its point contains the whole trajectory of this dynamical system.

Example

Consider the linear system

$$\dot{\xi} = -\xi,\ \xi \in \mathbb{R}^{n_s},\ \dot{\eta} = \eta,\ \eta \in \mathbb{R}^{n_u}$$

It has two invariant planes: plane $\{\eta = 0\}$ is called the stable plane, and the plane $\{\xi = 0\}$ is called the unstable plane; trajectories in these planes exponentially fast tend to the origin of the coordinates as $t \to +\infty$ and $t \to -\infty$ respectively. Similarly, one can define the stable and the unstable planes T^s and T^u for any hyperbolic linear system.

Stable and unstable manifolds of hyperbolic equilibria, periodic trajectories and fixed points

Consider an ODE

$$\dot{x} = Ax + O(|x|^2),\ x \in \mathbb{R}^n$$

with the right hand side of smoothness C^r (here one can consider also $r = \infty$ and $r = \omega$; the last notation is for the class of analytic functions). Assume that the matrix A has n_s and n_u eigenvalues in the left and right complex half-plane respectively, $n_s + n_u = n$. Denote T^s and T^u the corresponding invariant planes of A.

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Theorem (The Hadamard-Perron theorem)

In some neighborhood of the origin this ODE has two C^r-smooth invariant manifolds W^s and W^u, which are tangent at the origin to the planes T^s and T^u respectively. Trajectories in these manifolds exponentially fast tend to the origin of the coordinates as $t \to +\infty$ and $t \to -\infty$ respectively. W^s and W^u are called stable and unstable manifolds of the equilibrium 0 respectively.

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For analytic systems this result was obtained by A.M.Lyapunov and A.Poincaré.

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LECTURE 9

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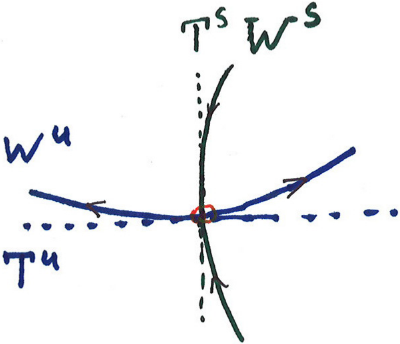

The center manifold theorem

Consider an ODE

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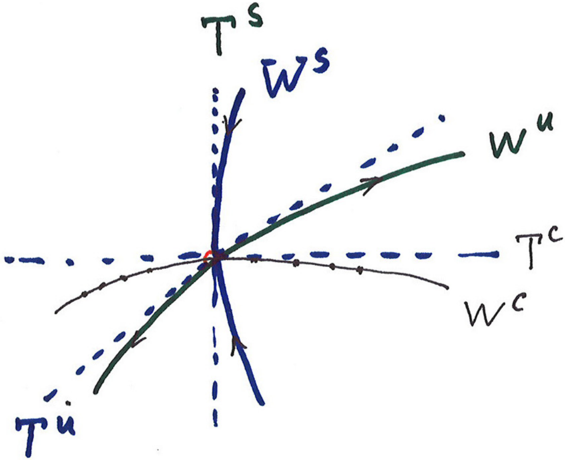
Remark

Behavior of trajectories on W^c is determined by nonlinear terms.

The center manifold theorem, continued



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The center manifold theorem, examples

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$$\dot{x} = xz - x^3, \ \dot{y} = y + x^4, \ \dot{z} = 0$$

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Theorem (Center manifold reduction: Pliss-Kelley-Hirsch-Pugh-Shub)

In a neighborhood of the coordinate origin this ODE is topologically equivalent to the direct product of restriction of this equation to the center manifold and the "standard saddle":

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The Tailor expansion for a center manifold can be computed by the method of undetermined coefficients.

Example: center manifold reduction

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Consider the system

$$\dot{x} = ax^2 + xb^T y + cx^3 + \ldots, \; x \in \mathbb{R}^1, \; \dot{y} = By + dx^2 + \ldots, \; y \in \mathbb{R}^m.$$

where B is a matrix without imaginary eigenvalues.

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Reduced onto the center manifold equation is

$$\dot{x} = ax^2 + (c - b^T Bd)x^3 + O(x^4).$$

Consider an ODE (actually, k-parametric family of ODE's)

$$\dot{x} = v(x, \alpha),\ v = A(\alpha)x + O(|x|^2),\ x \in \mathbb{R}^n,\ \alpha \in \mathbb{R}^k$$

with the right hand side of smoothness C^2. Assume that the matrix $A(0)$ has $n_s,\ n_u$ and n_c eigenvalues in the left complex half-plane, right complex half-plane and on imaginary axis respectively, $n_s + n_u + n_c = n$.

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Consider the extended system

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This system has in a neighborhood of the origin of the coordinates (x, α) a center manifold of dimension $n_c + k$.

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The restriction of the original family ODE's to the central manifold of the extended system is called *the reduced family*. According to the reduction principle in the problem of the topological classification without loss of generality one may consider only reduced families.

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If family of dynamical systems is *generic*, then *the codimension* of a bifurcation is the difference between the dimension of the parameter space and the dimension of the corresponding bifurcation boundary.

Some definitions in bifurcation theory

The phase portrait of a dynamical system is a partitioning of the state space into orbits.

Consider a dynamical system that depends on parameters (actually, family of dynamical systems).

The appearance of a topologically nonequivalent phase portraits under variation of parameters is called *a bifurcation*.

A bifurcation diagram of the dynamical system is a stratification of its parameter space induced by the topological equivalence, together with representative phase portraits for each stratum.

A bifurcation boundary is a surface in the parameter space on which a bifurcation occurs.

If family of dynamical systems is *generic*, then *the codimension* of a bifurcation is the difference between the dimension of the parameter space and the dimension of the corresponding bifurcation boundary. The codimension of the bifurcation of a given type is the minimal number of parameters of families in which that bifurcation type occurs. Equivalently, the codimension is the number of equality conditions that characterize a bifurcation.

Exercises

1. Consider the system

$$\dot{x} = xz - x^3, \ \dot{y} = y + x^4, \ \dot{z} = 0$$

Show that this system has a center manifold given by $y = V(x, z)$, where V is a C^6 function in x if $|z| < 1/6$, but only a C^4 function in x for $|z| < 1/4$.

2. Consider the system

$$\dot{x} = ax^2 + xb^T y + cx^3 + \ldots, \ \ x \in \mathbb{R}^1, \ \dot{y} = By + dx^2 + px^3 + xq^T y + \ldots, \ \ y \in \mathbb{R}^m.$$

where B is a matrix without imaginary eigenvalues. Find center manifold of 0 with accuracy $O(x^4)$.

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