Lectures on Dynamical Systems

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Part 2

LECTURE 10

LOCAL INVARIANT MANIFOLDS

Consider an ODE

$$\dot{x} = Ax + O(|x|^2), \; x \in \mathbb{R}^n$$

with the right hand side of smoothness C^r, $r < \infty$. Assume that the matrix A has n_s, n_u and n_c eigenvalues in the left complex half-plane, right complex half-plane and on imaginary axis respectively, $n_s + n_u + n_c = n$. Denote T^s, T^u and T^c the corresponding invariant planes of A . (Note: "s" is for "stable", "u" is for "unstable", "c" is for "center ").

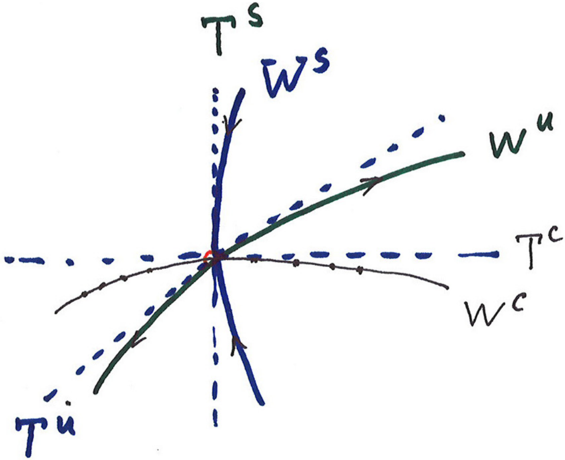
Theorem (The center manifold theorem: Pliss-Kelley-Hirsch-Pugh-Shub)

In some neighborhood U of the origin this ODE has C^r-smooth invariant manifolds W^s, W^u and C^{r-1}-smooth invariant manifold W^c, which are tangent at the origin to the planes T^s, T^u and T^c respectively. Trajectories in the manifolds W^s and W^u exponentially fast tend to the origin as $t \to +\infty$ and $t \to -\infty$ respectively. Trajectories which remain in U for all $t \geq 0$ ($t \leq 0$) tend to W^c as $t \to +\infty$ ($t \to -\infty$). W^s , W^u and W^c are called the stable, the unstable and a center manifolds of the equilibrium 0 respectively.

Remark

Behavior of trajectories on W^c is determined by nonlinear terms.

The center manifold theorem, continued



Remark

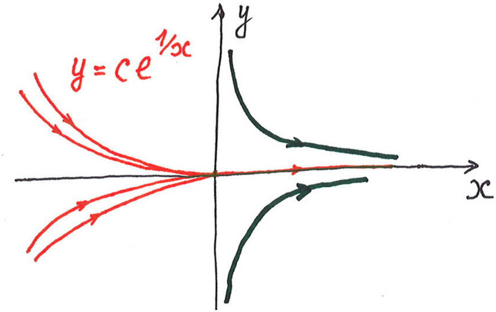
If the original equation has smoothness C^∞ or C^ω, then W^s and W^u also have smoothness C^∞ or C^ω. However W^c in general has only a finite smoothness.

Remark

If $n_s = 0$ or $n_u = 0$ and the original equation has smoothness C^r, $r < \infty$, then W^c has smoothness C^r.

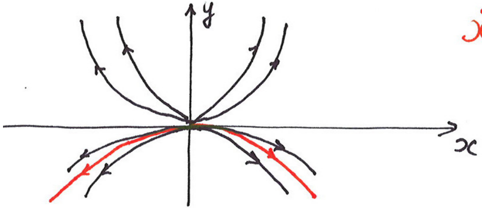
Example (A center manifold need not be unique)

$$\dot{x} = x^2, \; \dot{y} = -y$$



Example (A center manifold in general has only finite smoothness)

$$\dot{x} = xz - x^3,\ \dot{y} = y + x^4,\ \dot{z} = 0$$



$\dot{x} = x z,\ \dot{y} = y, z > 0$

$y = c|x|^{1/z}$

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with the right hand side of smoothness C^2. Assume that the matrix A has n_s, n_u and n_c eigenvalues in the left complex half-plane, right complex half-plane and on imaginary axis respectively, $n_s + n_u + n_c = n$.

Theorem (Center manifold reduction: Pliss-Kelley-Hirsch-Pugh-Shub)

In a neighborhood of the coordinate origin this ODE is topologically equivalent to the direct product of restriction of this equation to the center manifold and the "standard saddle":

$$\dot{\kappa} = w(\kappa), \ \kappa \in W^c, \ \dot{\xi} = -\xi, \ \xi \in \mathbb{R}^{n_s}, \ \dot{\eta} = \eta, \ \eta \in \mathbb{R}^{n_u}$$

The Tailor expansion for a center manifold can be computed by the method of undetermined coefficients.

Center manifold reduction, continued

Consider an ODE

$$\dot{x} = Ax + O(|x|^2), \; x \in \mathbb{R}^n$$

with the right hand side of smoothness C^r, $r > 2$. Assume that the matrix A is block-diagonal with blocks B and C, where B is $n_c \times n_c$-matrix with all eigenvalues on imaginary axis and B is $n_s \times n_s$-matrix with all eigenvalues in the left complex half-plane, $n_c + n_s = n$.

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Theorem (Reduction near a center manifold)

In a neighborhood of the coordinate origin this ODE by a C^{r-1}-smooth transformation of variables $x \mapsto \kappa, \xi$ which is C^1-close to the identity near the origin the system can be reduced to the form

$$\begin{aligned}\dot{\kappa} &= B\kappa + G(\kappa),\ \ \kappa \in \mathbb{R}^{n_c}, \\ \dot{\xi} &= (C + F(\kappa, \xi))\xi,\ \ \xi \in \mathbb{R}^{n_s},\end{aligned}$$

where $G \in \mathbb{C}^r,\ F \in C^{r-1},\ G(0) = 0,\ \partial G(0)/\partial\kappa = 0,\ F(0,0) = 0.$

Remark

The surface $\{\xi = 0\}$ is a center manifold.

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Remark

The surface $\{\xi = 0\}$ is a center manifold.

The Tailor expansion for the transformation $x \mapsto \kappa, \xi$ can be computed by the methods of normal forms theory.

Consider an ODE (actually, k-parametric family of ODE's)

$$\dot{x} = v(x, \alpha),\ v = A(\alpha)x + O(|x|^2),\ x \in \mathbb{R}^n,\ \alpha \in \mathbb{R}^k$$

with the right hand side of smoothness C^2. Assume that the matrix $A(0)$ has n_s, n_u and n_c eigenvalues in the left complex half-plane, right complex half-plane and on imaginary axis respectively, $n_s + n_u + n_c = n$.

Consider the extended system

$$\dot{x} = v(x, \alpha),\ \dot{\alpha} = 0$$

This system has in a neighborhood of the origin of the coordinates (x, α) a center manifold of dimension $n_c + k$.

Theorem (Shoshitaishvili reduction principle)

In a neighborhood of the coordinates' origin this ODE is topologically equivalent to the direct product of restriction of this equation to the center manifold and the "standard saddle":

$$\dot{\varkappa} = w(\varkappa, \alpha),\ \varkappa \in \mathbb{R}^{n_c},\ \dot{\alpha} = 0,\ \alpha \in \mathbb{R}^k,\ \dot{\xi} = -\xi,\ \xi \in \mathbb{R}^{n_s},\ \dot{\eta} = \eta,\ \eta \in \mathbb{R}^{n_u}$$

The homeomorphism which realizes equivalence does not change α.

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If family of dynamical systems is *generic*, then *the codimension* of a bifurcation is the difference between the dimension of the parameter space and the dimension of the corresponding bifurcation boundary. The codimension of the bifurcation of a given type is the minimal number of parameters of families in which that bifurcation type occurs. Equivalently, the codimension is the number of equality conditions that characterize a bifurcation.

Example (Saddle-node bifurcation; also called fold, tangent, limit point, turning point bifurcation)

The saddle-node bifurcation is a local bifurcation which takes place in generic ODEs when at some value of a parameter there is an equilibrium with the eigenvalue 0. In this case as the parameter changes two equilibria collide and disappear.

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The saddle-node bifurcation is a local bifurcation which takes place in generic ODEs when at some value of a parameter there is an equilibrium with the eigenvalue 0. In this case as the parameter changes two equilibria collide and disappear. Consider a one-parametric family of ODEs

$$\dot{x} = ax^2 + \alpha + O(|x|^3 + |\alpha x| + \alpha^2 + y^2 + \ldots), \ \dot{y} = -y + O(y^2 + |\alpha| + x^2 + \ldots)$$

Here $x \in \mathbb{R}^1, \ y \in \mathbb{R}^1, \ \alpha \in \mathbb{R}^1, \ a = \mathrm{const} \neq 0$.

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On the local center manifold for the extended system $y = O(|\alpha| + x^2)$.

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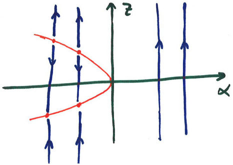
$$\dot{x} = ax^2 + \alpha + O(|x|^3 + |\alpha x| + \alpha^2),\ \dot{\alpha} = 0$$

The *truncated system* is

$$\dot{z} = az^2 + \alpha,\ \dot{\alpha} = 0$$

Example: Saddle-node bifurcation, continued

The phase portrait of the truncated system for $a > 0$ looks like this:



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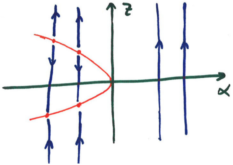

In the reduced on the central manifold family as the parameter α groves and passes trough 0 two equilibria, stable and unstable ones, collide and disappear. One can drove the bifurcation diagram for this codimension 1 bifurcation:

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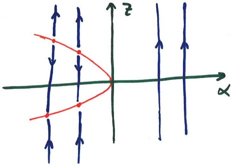

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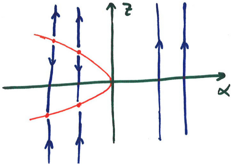

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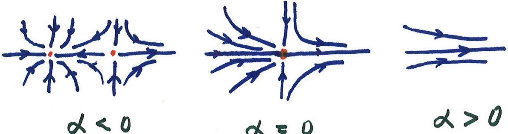
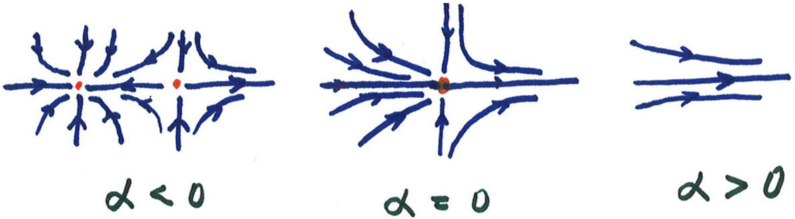
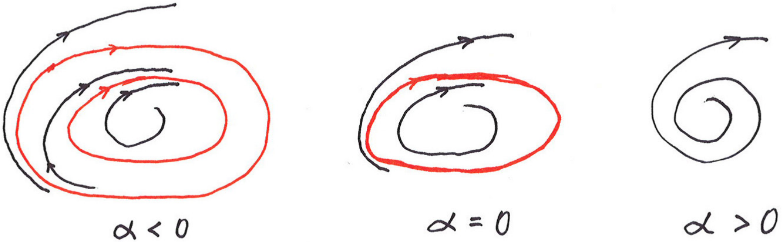
For the the original system the bifurcation diagram looks as follows:



Analogous bifurcation, also called the saddle-node bifurcation, takes place in generic ODEs when at some value of a parameter there is a periodic trajectory with the multiplier 1 (and in generic maps when at some value of a parameter there is a fixed point with the multiplier 1). As parameter changes two periodic trajectory (respectively, two fixed points) collide and disappear. The bifurcation diagram looks as follows (for periodic trajectories this is a picture on Poincaré section):



The bifurcation diagram for a planar system looks as follows:



Example of non-local bifurcation: The bifurcation of a limit cycle from the homoclinic loop of the saddle-node

Consider an ODE

$$\dot{x} = v(x, \alpha), \ x \in \mathbb{R}^n, \ \alpha \in \mathbb{R}^k$$

Let for $\alpha = 0$ this equation have a fixed point with all eigenvalues in the left half-plane but one equal to 0 (a saddle-node).

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Assume that for $\alpha = 0$ there is a homoclinic trajectory to this saddle-node.

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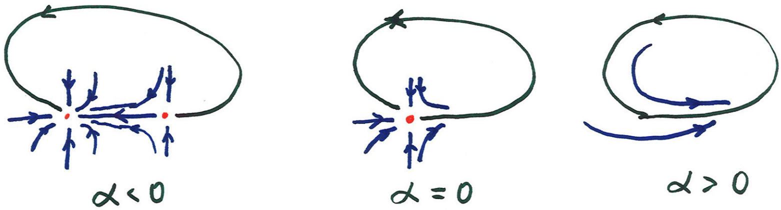
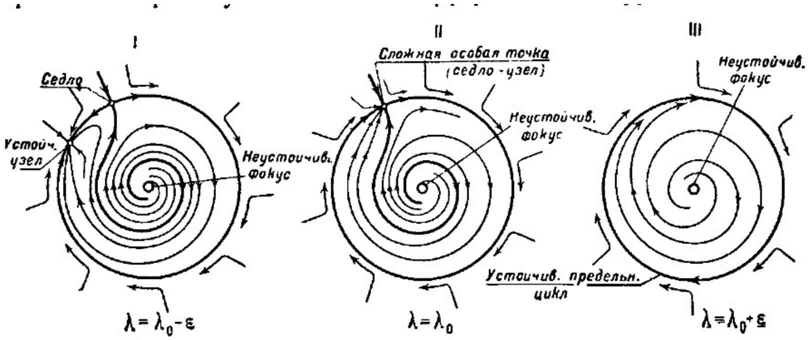
Let for $\alpha = 0$ this equation have a fixed point with all eigenvalues in the left half-plane but one equal to 0 (a saddle-node).
Assume that for $\alpha = 0$ there is a homoclinic trajectory to this saddle-node.
Under some generality assumptions the bifurcation diagram looks as follows (Andronov-Vitt-Leontovich-Shilnikov):



This is a codimension 1 bifurcation.

The bifurcation of a limit cycle from the homoclinic loop of the saddle-node, continued

The metamorphose of a phase portrait in 2D with such bifurcation may look like this (example from A.A.Andronov, A.A. Vitt, S.E. Khajkin, Theory of Oscillations, 1966):



NORMAL FORMS

Preliminary transformation: shift of the origin

Consider an ODE depending on parameters (actually, a family of ODEs)

$$\dot{x} = v(x, \alpha),\ x \in D \subset \mathbb{R}^n,\ \ \alpha \in U \subset \mathbb{R}^k,\ \ v \in C^2(D \times U)$$

Let for $\alpha = \alpha_0$ this ODE has an equilibrium $x = x_0$. Therefore,

$$\dot{x} = \frac{\partial v(x_0, \alpha_0)}{\partial x}(x - x_0) + O(|x - x_0|^2 + |\alpha - \alpha_0|)$$

Assume that the equilibrium is non-degenerate, i.e. matrix $A_0 = \frac{\partial v(x_0,\alpha_0)}{\partial x}$ is non-degenerate (does not have the eigenvalue 0). Then by the implicit function theorem for each value of α close enough to α_0 the equation has the equilibrium $x = X(\alpha)$ such that $X(\alpha_0) = x_0$. Introduce $\tilde{x} = x - X(\alpha)$. We get the ODE whose equilibrium is $\tilde{x} = 0$ for all values of α under consideration.

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In the following we will assume that there is no eigenvalue 0. So, without loss of generality we may assume that the equilibrium is at the coordinate origin.

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In the following we will assume that there is no eigenvalue 0. So, without loss of generality we may assume that the equilibrium is at the coordinate origin.

Recall that if there is the eigenvalue 0, then typically there is saddle-node bifurcation of equilibria.

Preliminary transformation: shift of the origin, continued

Consider a map depending on parameters (actually, a family of maps)

$$x \mapsto P(x,\alpha),\; x \in D \subset \mathbb{R}^n,\; \alpha \in U \subset \mathbb{R}^k,\; P \in C^2(D \times U)$$

Let for $\alpha = \alpha_0$ this map has a fixed point $x = x_0$. Therefore,

$$x \mapsto x_0 + \frac{\partial P(x_0,\alpha_0)}{\partial x}(x - x_0) + O(|x - x_0|^2 + |\alpha - \alpha_0|)$$

Assume that the fixed point is non-degenerate, i.e. it does not have the multiplier 1 (i.e. matrix $A_0 = \frac{\partial P(x_0,\alpha_0)}{\partial x}$ does not have the eigenvalue 1). Then by the implicit function theorem for each value of α close enough to α_0 the map has the fixed point $x = X(\alpha)$ such that $X(\alpha_0) = x_0$. Introduce $\tilde{x} = x - X(\alpha)$. We get the map whose fixed point is $\tilde{x} = 0$ for all values of α under consideration.

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Let for $\alpha = \alpha_0$ this map has a fixed point $x = x_0$. Therefore,

$$x \mapsto x_0 + \frac{\partial P(x_0,\alpha_0)}{\partial x}(x - x_0) + O(|x - x_0|^2 + |\alpha - \alpha_0|)$$

Assume that the fixed point is non-degenerate, i.e. it does not have the multiplier 1 (i.e. matrix $A_0 = \frac{\partial P(x_0,\alpha_0)}{\partial x}$ does not have the eigenvalue 1). Then by the implicit function theorem for each value of α close enough to α_0 the map has the fixed point $x = X(\alpha)$ such that $X(\alpha_0) = x_0$. Introduce $\tilde{x} = x - X(\alpha)$. We get the map whose fixed point is $\tilde{x} = 0$ for all values of α under consideration.

In the following we will assume that there is no multiplier 1. So, without loss of generality we may assume that the fixed point is at the coordinate origin.

Recall that if there is the multiplier 1, then typically there is saddle-node bifurcation of fixed points.

Preliminary transformation: shift of the origin, continued

Consider an ODE depending on parameters (actually, a family of ODEs)

$$\dot{x} = v(x, \alpha),\ x \in D \subset \mathbb{R}^n,\ \alpha \in U \subset \mathbb{R}^k,\ v \in C^2(D \times U)$$

Let for $\alpha = \alpha_0$ this ODE has a periodic trajectory.

Preliminary transformation: shift of the origin, continued

Consider an ODE depending on parameters (actually, a family of ODEs)

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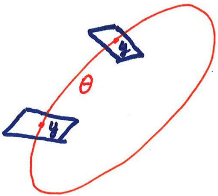

In the normal coordinates near this trajectory the equation has the form

$$\frac{dy}{d\theta} = w(y, \theta, \alpha),\ \ w(0, \theta, \alpha_0) \equiv 0,\ \ y \in \mathbb{R}^{n-1},\ \ \theta \in \mathbb{S}^1$$

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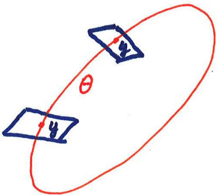

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The monodromy map for the section $\{\theta = 0\}$ has for $\alpha = \alpha_0$ the fixed point at $y = 0$. Assume that the periodic trajectory is non-degenerate, i.e. the fixed point does not have multiplier 1. Then for each value of α close enough to α_0 the map has the fixed point $y = y_*(\alpha)$ such that $y_*(\alpha_0) = 0$. The equation has periodic solution $Y(\theta, \alpha),\ \theta \in \mathbb{S}^1$ with the initial condition $Y(0, \alpha) = y_*(\alpha)$.

Preliminary transformation: shift of the origin, continued

Introduce $\tilde{y} = y - Y(\theta, \alpha)$. We get the time-periodic ODE which has equilibrium $\tilde{y} = 0$ for all values of α under consideration.

In the following we will assume that there is no multiplier 1. So, without loss of generality we may assume that the system in normal coordinates has equilibrium at the coordinate origin.

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In the following we will assume that there is no multiplier 1. So, without loss of generality we may assume that the system in normal coordinates has equilibrium at the coordinate origin.

If all multipliers are different, then according to Floquet-Lyapunov theory without loss of generality we may assume that the linearised near the equilibrium system has constant coefficients.

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LECTURE 11

NORMAL FORMS

Consider an ODE

$$\dot{x} = Ax + O(|x|^2), \ x \in \mathbb{R}^n$$

where A is a linear operator. Assume that right hand side of this ODE is analytic in some neighborhood of 0.
Denote $\lambda_1, \lambda_2, \dots, \lambda_n$ the eigenvalues of A.

Definition

The set of eigenvalues of the operator A is called *a resonant* one if a relation of the form

$$\lambda_s = m_1\lambda_1 + m_2\lambda_2 + \ldots + m_n\lambda_n,$$

with integer non-negative $m_1, m_2, \dots, m_n$ such that $\sum\limits_{j=1}^{n} m_j \geq 2$ is satisfied. This relation is called *a resonance relation* or just *a resonance*. The value $|m| = \sum\limits_{j=1}^{n} m_j$ is called *an order of the resonance*.

Denote $(m, \lambda) \stackrel{\text{def}}{=} m_1\lambda_1 + m_2\lambda_2 + \ldots + m_n\lambda_n$.

Resonances near equilibria

Consider an ODE

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Denote $(m, \lambda) \stackrel{\text{def}}{=} m_1\lambda_1 + m_2\lambda_2 + \ldots + m_n\lambda_n$.

Example

$\lambda_1 = \lambda_2 + \lambda_3$ is the resonance of order 2. $2\lambda_1 = 3\lambda_2$ is not a resonance.
If $\lambda_1 = -\lambda_2$ then there is infinite number of resonances $\lambda_s = \lambda_s + k(\lambda_1 + \lambda_2)$, $k = 1, 2, 3, \ldots$.

Reduction to a linear system, the Poincaré theorem

Theorem (H. Poincaré)

If eigenvalues of an equilibrium do not satisfy resonance relations up to an order N inclusively, then by a polynomial real close to the identical transformation of variables

$$x = y + O(|y|^2)$$

the system is reducible to the form

$$\dot{y} = Ay + O(|y|^{N+1})$$

Corollary

If there are no resonances of any order, then a formal transformation of variables reduces original nonlinear system to the linear one.

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Corollary

If there are no resonances of any order, then a formal transformation of variables reduces original nonlinear system to the linear one.

If all eigenvalues are situated in one complex half-plane, either in the left or in the right one, then the formal series for the transformation of variables converges in some neighborhood of 0. So, by an analytic transformation of variables the system is reducible to the linear one (H. Poincaré).

Proof of the Poincaré theorem

The system under consideration has the form

$$\dot{x} = Ax + V(x),\ \ V(x) = v_2(x) + v_3(x) + \ldots + v_N(x) + O(|x|^{N+1})$$

where $v_r(x)$ is the homogeneous vector polynomial of x of degree r.

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We are looking for a transformation of variables $x \mapsto y$ of the form

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$$\dot{y} + \frac{\partial h}{\partial y}\dot{y} = A(y + h(y)) + V(y + h(y)) + O(|y|^{N+1})$$

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Assume that the transformation reduces the system to the required form. Equating terms of order r we get *a homological equation* (called also *a co-homological equation*)

$$\frac{\partial h_r}{\partial y}Ay - Ah_r(y) = V_r(y)$$

where V_r is the homogeneous vector polynomial of degree r whose coefficients are expressed through coefficients of $v_2, \ldots, v_r, \; h_2, \ldots, h_{r-1}$.

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Lemma

In absence of resonances of order r for any V_r the homological equation has a unique solution h_r.

Proof of the Poincaré theorem

The system under consideration has the form

$$\dot{x} = Ax + V(x),\;\; V(x) = v_2(x) + v_3(x) + \ldots + v_N(x) + O(|x|^{N+1})$$

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where V_r is the homogeneous vector polynomial of degree r whose coefficients are expressed through coefficients of $v_2, \ldots, v_r,\; h_2, \ldots, h_{r-1}$.

Lemma

In absence of resonances of order r for any V_r the homological equation has a unique solution h_r.

Induction in r completes the proof.

To simplify the reasoning assume that eigenvalues of A are all different (the result is valid in the general case). The homological equation has the form

$$\frac{\partial h(y)}{\partial y}Ay - Ah(y) = U(y)$$

(Note that $(\partial h/\partial y)Ay - Ah$ is the commutator of the vector fields Ay and h.) Here $U(y)$, $h(y)$ are the homogeneous vector polynomials of y of degree r. Let $e_1, e_2, \ldots, e_n$ be eigenvectors of the complexified operator A, that correspond to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. The eigenvectors form a basis in $\mathbb{C}^n$. Let $y_1, y_2, \ldots, y_n$ be coordinates of y in this basis. Then

$$U = \sum_{s=1,\ldots,n;\ |m|=r} U_{s,m} y^m e_s,\ \ h = \sum_{s=1,\ldots,n;\ |m|=r} h_{s,m} y^m e_s$$

Here
$m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$, $m_i \geq 0$, $|m| \stackrel{\text{def}}{=} m_1 + \ldots + m_n$, $y^m \stackrel{\text{def}}{=} y_1^{m_1} y_2^{m_2} \ldots y_n^{m_n}$.
Equating in the homological equation the coefficients in front of $y^m e_s$, we get

$$(m_1\lambda_1 + m_2\lambda_2 + \ldots + m_n\lambda_n - \lambda_s)h_{s,m} = U_{s,m}$$

Thus, $h_{s,m} = U_{s,m}/((m,\lambda) - \lambda_s)$. If y is real, then $h(y)$ is real. This completes the proof.

Resonant monomials, resonant normal form near equilibria

For simplicity of formulations assume that eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the operator A are all different. So, the the eigenvectors $e_1, e_2, \ldots, e_n$ of the complexified operator A form a basis in $\mathbb{C}^n$.

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Let some system $\mathcal{S}$ of resonance relations be given. We will assume that $\mathcal{S}$ contains all resonance relations which can be derived from any subsystem of $\mathcal{S}$.

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Definition

A vector monomial $x^m e_s$ is called a *resonant* one for resonances in $\mathcal{S}$ if the resonance relation $\lambda_s = (m, \lambda)$ is presented in the system $\mathcal{S}$.

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Example

If $\mathcal{S}$ includes relation $\lambda_1 = \lambda_2 + \lambda_3$, then the vector monomial $x_2 x_3 e_1$ is a resonant one. If $\mathcal{S}$ includes relation $\lambda_1 = 2\lambda_1 + \lambda_2$, then all vector monomials $(x_1 x_2)^k x_s e_s$ are resonant ones.

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Definition

A system

$$\dot{x} = Ax + \dots$$

is said to be in the resonant normal form for resonances from $\mathcal{S}$ if the nonlinear part of its right hand side is a sum of resonant vector monomials.

Reduction to resonant normal form, the Poincaré-Dulac theorem

Theorem (H. Poincaré-H.Dulac)

If eigenvalues of an equilibrium do not satisfy resonance relations up to an order N inclusively except, may be, resonances from $\mathcal{S}$, then by a polynomial real close to the identical transformation of variables

$$x = y + O(|y|^2)$$

the system is reducible to the form

$$\dot{y} = Ay + w(y) + O(|y|^{N+1})$$

were w is a sum of resonant vector monomials of degrees not exceeding N.

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Corollary

If there are no resonances of any order, except, may be, resonances from $\mathcal{S}$, then a formal transformation of variables reduces original system to a system in a formal resonant normal form.

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Corollary

If there are no resonances of any order, except, may be, resonances from $\mathcal{S}$, then a formal transformation of variables reduces original system to a system in a formal resonant normal form.

Example

If $n = 2$ and the only possible resonance is $\lambda_1 = 2\lambda_2$, then the system in formal normal form is $\dot{x}_1 = \lambda_1 x_1 + c x_2^2,\ \dot{x}_2 = \lambda_2 x_2,\ c$=const.

Proof of the Poincaré-Dulac theorem

The system under consideration has the form

$$\dot{x} = Ax + V(x), \;\; V(x) = v_2(x) + v_3(x) + \ldots + v_N(x) + O(|x|^{N+1})$$

where $v_r(x)$ is the homogeneous vector polynomial of x of degree r.

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where $v_r(x)$ is the homogeneous vector polynomial of x of degree r.
We are looking for a transformation of variables $x \mapsto y$ of the form

$$x = y + h(y), \;\; h(y) = h_2(y) + h_3(y) + \ldots + h_N(y)$$

which reduces the system to the form

$$\dot{y} = Ay + w(y) + O(|y|^{N+1}), \;\; w(y) = w_2(y) + w_3(y) + \ldots + w_N(y)$$

where $h_r(y), w_r(y)$ are homogeneous vector polynomials of y of degree r, and $w_r(y)$ contains only resonant monomials.

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where $h_r(y), w_r(y)$ are homogeneous vector polynomials of y of degree r, and $w_r(y)$ contains only resonant monomials. We have

$$\dot{y} + \frac{\partial h}{\partial y}\dot{y} = A(y + h(y)) + V(y + h(y)) + O(|y|^{N+1})$$

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$$\dot{y} + \frac{\partial h}{\partial y}\dot{y} = A(y + h(y)) + V(y + h(y)) + O(|y|^{N+1})$$

Assume that the transformation reduces the system to the required form. Equating terms of order r we get *a homological equation*

$$\frac{\partial h_r}{\partial y}Ay - Ah_r(y) = V_r(y) - w_r(y)$$

where V_r is the homogeneous vector polynomial of degree r whose coefficients are expressed through coefficients of $v_2, \ldots, v_r,\ h_2, \ldots, h_{r-1},\ w_2, \ldots, w_{r-1}$.

Proof of the Poincaré-Dulac theorem

The system under consideration has the form

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where $h_r(y), w_r(y)$ are homogeneous vector polynomials of y of degree r, and $w_r(y)$ contains only resonant monomials. We have

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Assume that the transformation reduces the system to the required form. Equating terms of order r we get *a homological equation*

$$\frac{\partial h_r}{\partial y}Ay - Ah_r(y) = V_r(y) - w_r(y)$$

where V_r is the homogeneous vector polynomial of degree r whose coefficients are expressed through coefficients of $v_2, \ldots, v_r,\ h_2, \ldots, h_{r-1},\ w_2, \ldots, w_{r-1}$.
Take as $w_r(y)$ the sum of resonant monomials in $V_r(y)$.

Lemma

For this choice of w_r the homological equation has a solution h_r in the form of the sum of non-resonant monomials. The solution in such form is a unique.

Proof of the Poincaré-Dulac theorem, continued

Lemma

For this choice of w_r the homological equation has a solution h_r in the form of the sum of non-resonant monomials. The solution in such form is a unique.

Induction in r completes the proof of the theorem.

Proof of the Poincaré-Dulac theorem, continued

Lemma

For this choice of w_r the homological equation has a solution h_r in the form of the sum of non-resonant monomials. The solution in such form is a unique.

Induction in r completes the proof of the theorem.

Proof of the Lemma about the homological equation.

The solution is constructed by the method of undetermined coefficients exactly as in the proof of the Poincaré theorem. Denominators in the formulas do not vanish because the right hand side of the homological equation does not contain resonant monomials.

□

Exercises

1. Check that the vector field $(\partial h/\partial y)Ay - Ah$ is the commutator of the vector fields Ay and h

2. The operator $(\partial(\cdot)/\partial y)Ay - A(\cdot)$ is a linear operator in the space of homogeneous vector polynomials of any given degree. Find eigenvalues of this operator.

3. Find formal normal form for a system of 3 equations in the case of the resonance $\lambda_1 = \lambda_2 + \lambda_3$.

4.Prove that the phase flow of any system in normal form for resonances in $\mathcal{S}$ commutes with the phase flow of its linear part provided that all resonance relation from $\mathcal{S}$ are indeed satisfied.

LECTURE 12

NORMAL FORMS

Consider an ODE

$$\dot{x} = Ax + O(|x|^2), \;\; x \in \mathbb{R}^n$$

where A is a linear operator. Assume that right hand side of this ODE is analytic in some neighborhood of 0.
Denote $\lambda_1, \lambda_2, \ldots, \lambda_n$ the eigenvalues of A.

Definition

The set of eigenvalues of the operator A is called *a resonant* one if a relation of the form

$$\lambda_s = m_1\lambda_1 + m_2\lambda_2 + \ldots + m_n\lambda_n,$$

with integer non-negative $m_1, m_2, \ldots, m_n$ such that $\sum_{j=1}^{n} m_j \geq 2$ is satisfied. This relation is called *a resonance relation* or just *a resonance*. The value $|m| = \sum_{j=1}^{n} m_j$ is called *an order of the resonance*.

Denote $(m, \lambda) \stackrel{\text{def}}{=} m_1\lambda_1 + m_2\lambda_2 + \ldots + m_n\lambda_n$.

Example

$\lambda_1 = \lambda_2 + \lambda_3$ is the resonance of order 2. $2\lambda_1 = 3\lambda_2$ is not a resonance.
If $\lambda_1 = -\lambda_2$ then there is infinite number of resonances $\lambda_s = \lambda_s + k(\lambda_1 + \lambda_2)$, $k = 1, 2, 3, \ldots$.

Resonant monomials, resonant normal form near equilibria

For simplicity of formulations assume that eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the operator A are all different. So, the the eigenvectors $e_1, e_2, \ldots, e_n$ of the complexified operator A form a basis in $\mathbb{C}^n$.

Let some system $\mathcal{S}$ of resonance relations be given. We will assume that $\mathcal{S}$ contains all resonance relations which can be derived from any subsystem of $\mathcal{S}$.

Definition

A vector monomial $x^m e_s$ is called a *resonant* one for resonances in $\mathcal{S}$ if the resonance relation $\lambda_s = (m, \lambda)$ is presented in the system $\mathcal{S}$.

Example

If $\mathcal{S}$ includes relation $\lambda_1 = \lambda_2 + \lambda_3$, then the vector monomial $x_2 x_3 e_1$ is a resonant one. If $\mathcal{S}$ includes relation $\lambda_1 = 2\lambda_1 + \lambda_2$, then all vector monomials $(x_1 x_2)^k x_s e_s$ are resonant ones.

Definition

A system

$$\dot{x} = Ax + \ldots$$

is said to be in the resonant normal form for resonances from $\mathcal{S}$ if the nonlinear part of its right hand side is a sum of resonant vector monomials.

Reduction to resonant normal form, the Poincaré-Dulac theorem

Theorem (H. Poincaré-H.Dulac)

If eigenvalues of an equilibrium do not satisfy resonance relations up to an order N *inclusively except, may be, resonances from* $\mathcal{S}$*, then by a polynomial real close to the identical transformation of variables*

$$x = y + O(|y|^2)$$

the system is reducible to the form

$$\dot{y} = Ay + w(y) + O(|y|^{N+1})$$

were w *is a sum of resonant vector monomials of degrees not exceeding* N*.*

Thus, the system without the term $O(|y|^{N+1})$ (also called *a truncated system*) is in a resonant normal form.

Corollary

If there are no resonances of any order, except, may be, resonances from $\mathcal{S}$*, then a formal transformation of variables reduces original system to a system in a formal resonant normal form.*

Example

If $n = 2$ and the only possible resonance is $\lambda_1 = 2\lambda_2$, then the system in formal normal form is $\dot{x}_1 = \lambda_1 x_1 + c x_2^2,\ \dot{x}_2 = \lambda_2 x_2,\ c$=const.

Example: normal form for Poincaré-Andronov-Hopf bifurcation

The Poincaré-Andronov-Hopf bifurcation is a local bifurcation which takes place in generic ODEs when an equilibrium loses stability as a pair of complex conjugate eigenvalues cross the imaginary axis of the complex plane.

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According to Poincaré-Dulac's theorem the system can be transformed to form

$$\begin{aligned}
\dot{z}_1 &= \lambda_1 z_1 + (c_0(z_1 z_2) + c_1(z_1 z_2)^2 + \ldots + c_{N-2}(z_1 z_2)^{(N-1)/2}) z_1 + O(|z|^{N+1}) \\
\dot{z}_2 &= \lambda_2 z_2 + (\bar{c}_0(z_1 z_2) + \bar{c}_1(z_1 z_2)^2 + \ldots + \bar{c}_{N-2}(z_1 z_2)^{(N-1)/2}) z_2 + \bar{O}(|z|^{N+1}) \\
\dot{z}_j &= \lambda_j z_j + (d_{j,0}(z_1 z_2) + d_{j,1}(z_1 z_2)^2 + \ldots + d_{j,N-2}(z_1 z_2)^{(N-1)/2}) z_j + O(|z|^{N+1}) \\
j &= 3, 4, \ldots, n
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\end{aligned}$$

The center manifold is approximated by the plane of variables z_1, z_2. For real initial data $z_2 = \bar{z}_1$ along solutions. Denote $z = z_1, \lambda = \lambda_1$. Truncated at the terms of the 3rd order equation for z is

$$\dot{z} = (\lambda + c_0|z|^2) z$$

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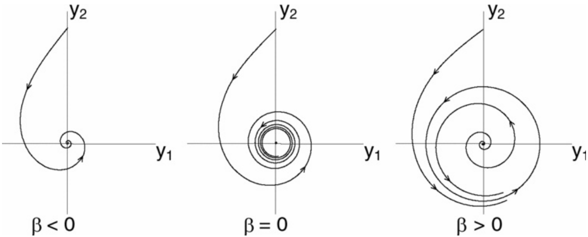
Introduce polar coordinates r, φ : $z = re^{i\varphi}$. We get equations:

$$\dot{r} = (\delta + ar^2)r, \ \dot{\varphi} = \omega + br^2, \text{ were } a = \mathrm{Re}\, c_0, \ b = \mathrm{Im}\, c_0$$

Example: normal form for Poincaré-Andronov-Hopf bifurcation, continued

The bifurcation diagram for the case $a < 0$ (so called *supercritical, or soft, or non-catastrophic bifurcation*) looks as follows (image by Yuri Kuznetsov at Scholarpedia, $\beta \equiv \delta = \mathrm{Re}\,\lambda$):



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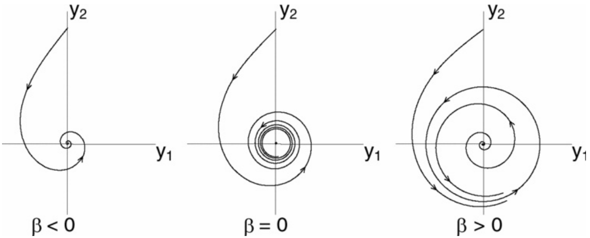
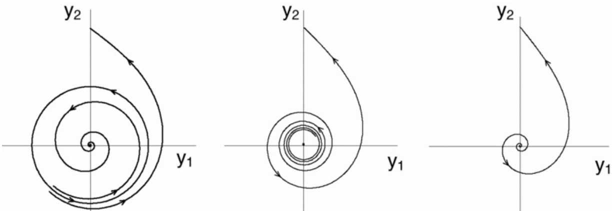
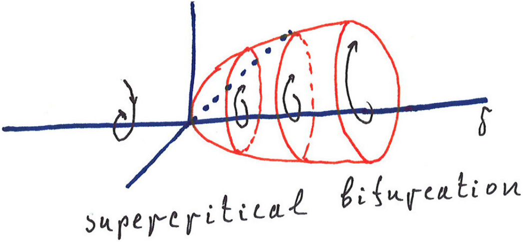


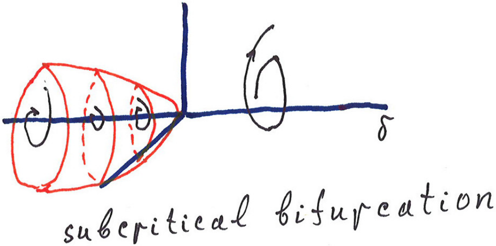
The bifurcation diagram for the case $a > 0$ (so called *subcritical, or sharp, or catastrophic bifurcation*) looks as follows (image by Yuri Kuznetsov at Scholarpedia, $\beta \equiv \delta = \mathrm{Re}\,\lambda$):



Example: normal form for Poincaré-Andronov-Hopf bifurcation, continued

The phase portraits for the extended system (we add equation $\dot{\delta} = 0$) looks as follows





Exercises

1. Consider the system

$$\dot{x} = -y + \delta x + \alpha xy, \;\; \dot{y} = x + \delta y + \beta xy + \gamma x^2$$

The parameter δ grows and passing through the value $\delta = 0$. For which values of parameters $\alpha,\ \beta, \gamma$ the stability loss of the equilibrium $x = y = 0$ will be a "soft" one?

Consider an ODE

$$\dot{x} = Ax + V(x,t),\ \ V(x,t+2\pi) = V(x,t),\ \ V = O(|x|^2),\ \ x \in \mathbb{R}^n$$

where A is a linear operator. Assume that function V is analytic in some neighborhood of $\{0\} \times \mathbb{S}^1$.
We use previous notation: $\lambda_j,\ j = 1,2,\ldots,n$ are eigenvalues of A,
$\lambda = (\lambda_1,\ \lambda_2,\ldots,\lambda_n)$, $m = (m_1, m_2, \ldots, m_n)$, $|m| = |m_1| + |m_2| + \ldots + |m_n|$,
$(m,\lambda) = m_1\lambda_1 + m_2\lambda_2 + \ldots + m_n\lambda_n$.

Definition

The set of eigenvalues of the operator A is called *a resonant* one if a relation of the form

$$\lambda_s = (m,\lambda) + ik,$$

is satisfied, where components of m are integer non-negative, $|m| \geq 2$, k is integer. This relation is called *a resonance relation* or just *a resonance*. The value $|m|$ is called *an order of the resonance*.

Note that number of resonances of given order $|m|$ is finite.

Assume that eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of the operator A are all different. So, the the eigenvectors $e_1, e_2, \dots, e_n$ of the complexified operator A form a basis in $\mathbb{C}^n$.

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A vector monomial $e^{ikt}x^m e_s$ is called a *resonant* one for resonances in $\mathcal{S}$ if the resonance relation $\lambda_s = (m, \lambda) + ik$ is presented in the system $\mathcal{S}$.

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A system

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is said to be in the resonant normal form for resonances from $\mathcal{S}$ if the nonlinear part of its right hand side is a sum of resonant vector monomials.

Reduction to resonant normal form near periodic trajectory

Theorem

If eigenvalues of an equilibrium of time-periodic system do not satisfy resonance relations up to an order N inclusively except, may be, resonances from $\mathcal{S}$, then by a polynomial in space coordinates and periodic in time real close to the identical transformation of variables

$$x = y + O(|y|^2)$$

the system is reducible to the form

$$\dot{y} = Ay + w(y, t) + O(|y|^{N+1})$$

were w is a sum of resonant vector monomials of degrees not exceeding N.

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Corollary

If there are no resonances of any order, except, may be, resonances from $\mathcal{S}$, then a formal transformation of variables reduces the original system to a system in a formal resonant normal form.

Reduction to resonant normal form near periodic trajectory, continued

Procedure of reduction to resonant normal form near a periodic trajectory is analogous to that near an equilibrium

The system under consideration has the form

$$\dot{x} = Ax + V(x,t), \;\; V(x,t) = v_2(x,t) + v_3(x,t) + \ldots + v_N(x,t) + O(|x|^{N+1})$$

where $v_r(x,t)$ is the homogeneous vector polynomial of x of degree r.

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where $v_r(x,t)$ is the homogeneous vector polynomial of x of degree r.

We are looking for a transformation of variables $x, t \mapsto y, t$ of the form

$$x = y + h(y,t),\ \ h(y,t) = h_2(y,t) + h_3(y,t) + \ldots + h_N(y,t)$$

which reduces the system to the form

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where $h_r(y,t), w_r(y,t)$ are homogeneous vector polynomials of y of degree r, and $w_r(y,t)$ contains only resonant monomials.

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where $h_r(y,t), w_r(y,t)$ are homogeneous vector polynomials of y of degree r, and $w_r(y,t)$ contains only resonant monomials. Plugging the transformation of variables into original differential equation, assuming that the transformed equation has required form and equating terms of order r we get *a homological equation*

$$\frac{\partial h_r}{\partial t} + \frac{\partial h_r}{\partial y} Ay - Ah_r(y,t) = V_r(y,t) - w_r(y,t)$$

where V_r is the homogeneous vector polynomial of degree r whose coefficients are expressed through coefficients of $v_2, \ldots, v_r,\ h_2, \ldots, h_{r-1},\ w_2, \ldots, w_{r-1}$.

Reduction to resonant normal form near periodic trajectory, continued

Procedure of reduction to resonant normal form near a periodic trajectory is analogous to that near an equilibrium

The system under consideration has the form

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which reduces the system to the form

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Take as $w_r(y,t)$ the sum of resonant monomials in $V_r(y,t)$.

Lemma

For this choice of w_r the homological equation has a solution h_r in the form of a sum of non-resonant monomials. The solution in such form is a unique.

Proof.

Let $e_1, e_2, \ldots, e_n$ be eigenvectors of the complexified operator A, that correspond to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. The eigenvalues of A are all different, and so the eigenvectors form a basis in $\mathbb{C}^n$. Let $y_1, y_2, \ldots, y_n$ be coordinates of y in this basis. Denote $U_r(y,t) = V_r(y,t) - w_r(y,t)$. Then

$$U_r = \sum_{k\in\mathbb{Z};\, s=1,\ldots,n;\, |m|=r} U_{k,s,m} e^{ikt} y^m e_s, \quad h = \sum_{k\in\mathbb{Z};\, s=1,\ldots,n;\, |m|=r} h_{k,s,m} e^{ikt} y^m e_s$$

Equating in the homological equation the coefficients in front of $e^{ik} y^m e_s$, we get

$$(ik + m_1\lambda_1 + m_2\lambda_2 + \ldots + m_n\lambda_n - \lambda_s) h_{s,m} = U_{s,m}$$

Thus, $h_{k,s,m} = U_{k,s,m}/(ik + (m,\lambda) - \lambda_s)$. If y is real, then $h(y)$ is real. This completes the proof.

□

Example: normal form for Neimark-Sacker bifurcation

The Neimark-Sacker bifurcation is a local bifurcation which takes place in generic ODEs when a periodic trajectory loses stability as a pair of complex conjugate multipliers cross the unit circle in the complex plane not close to points 1, -1, $e^{\pm i2\pi/3}$, $e^{\pm i\pi/2}$.

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Consider the case $n = 2$. General case reduces to this one by means of Shoshitaisvili reducton principle. Assume that $\lambda_{1,2} = \delta \pm i\omega$, where δ is small and $\omega \neq 0$.

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Consider the case $n = 2$. General case reduces to this one by means of Shoshitaisvili reducton principle. Assume that $\lambda_{1,2} = \delta \pm i\omega$, where δ is small and $\omega \neq 0$. The resonance relation

$$\lambda_1 = m_1\lambda_1 + m_2\lambda_2 + ik$$

for $\lambda_{1,2} = \pm i\omega$ reduces to

$$\omega(m_1 - m_2 - 1) + k = 0$$

The correspoding multiplier is $\rho = e^{2\pi i\omega}$.
Enumerate possible resonances of the 2nd and 3rd order:

$m_1 = 2,\ m_2 = 0,\ \omega = -k,\ \rho = 1$
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$m_1 = 3,\ m_2 = 0,\ 2\omega = -k,\ \rho = e^{-\pi ik} = \pm 1$
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Example: normal form for Neimark-Sacker bifurcation

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Consider the case $n = 2$. General case reduces to this one by means of Shoshitaisvili reducton principle. Assume that $\lambda_{1,2} = \delta \pm i\omega$, where δ is small and $\omega \neq 0$. The resonance relation

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Our assumptions about multipliers exclude all cases but $m_1 = 2,\ m_2 = 1$.

Example: normal form for Neimark-Sacker bifurcation, continued

So, the only possible resonance relation with $|m| \leq 3$ is $\lambda_1 = 2\lambda_1 + \lambda_2$, or $\lambda_1 + \lambda_2 = 0$ like for Poincaré-Andronov-Hopf bifurcation.

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$$\dot{z}_1 = \lambda_1 z_1 + c_0(z_1 z_2) z_1 + O(|z|^4), \ \dot{z}_2 = \lambda_2 z_2 + \bar{c}_0(z_1 z_2) z_2 + \bar{O}(|z|^4)$$

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For real initial data along solutions $z_2 = \bar{z}_1$. Denote $z = z_1, \lambda = \lambda_1$. Truncated at the terms of the 3rd order equation for z is

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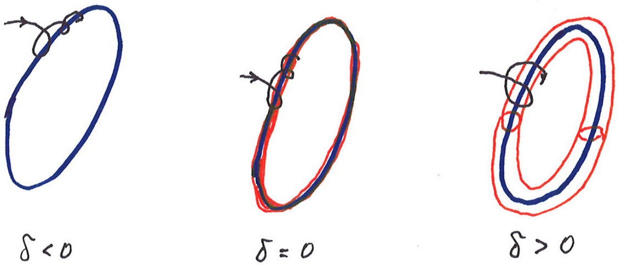
The bifurcation diagrams look like this: Supercritical case:



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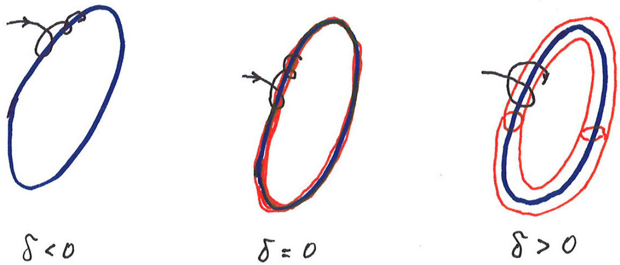

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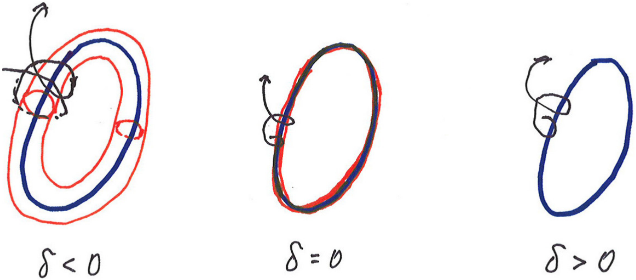
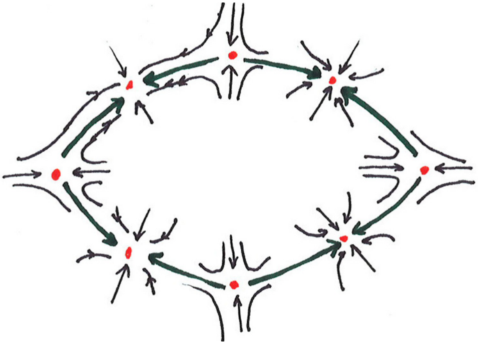


There is nevertheless essential difference in behavior of trajectories in exact and truncated system. The Poincaré section for exact system may look like this.



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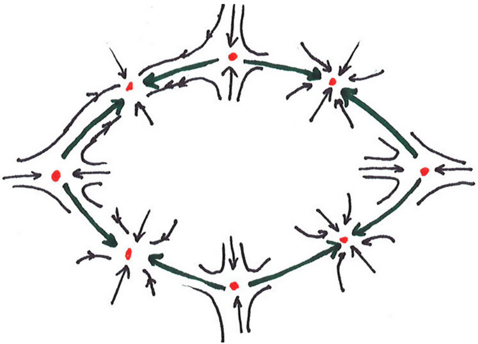

In the original system as a parameter changes on the invariant torus appear an disappear isolated periodic trajectories.

Invariant torus in general has only finite smoothness.

On stability loss of periodic trajectory near the resonance

What happens if a pair of complex-conjugated multipliers ρ, $\bar{\rho}$ cross the unit circle near the points $e^{\pm \frac{2\pi i}{3}}$?

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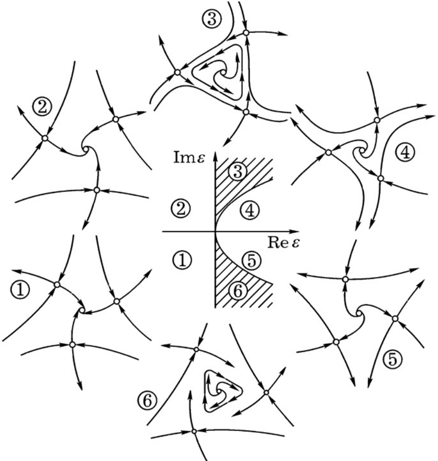

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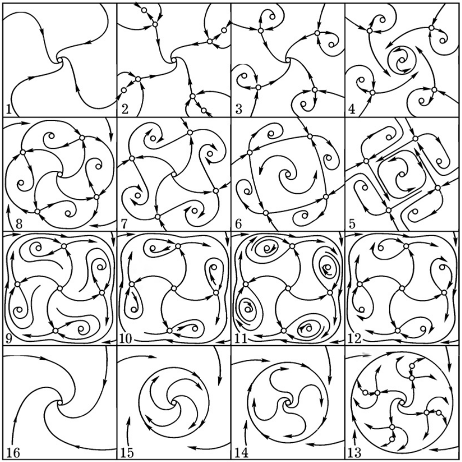

LECTURE 13

NORMAL FORMS

Resonances near periodic trajectory

Consider an ODE

$$\dot{x} = Ax + V(x,t),\ \ V(x,t+2\pi) = V(x,t),\ \ V = O(|x|^2),\ \ x \in \mathbb{R}^n$$

where A is a linear operator. Assume that function V is analytic in some neighborhood of $\{0\} \times \mathbb{S}^1$.
We use previous notation: λ_j, $j = 1, 2, \ldots, n$ are eigenvalues of A,
$\lambda = (\lambda_1,\, \lambda_2, \ldots, \lambda_n)$, $m = (m_1, m_2, \ldots, m_n)$, $|m| = |m_1| + |m_2| + \ldots + |m_n|$,
$(m, \lambda) = m_1\lambda_1 + m_2\lambda_2 + \ldots + m_n\lambda_n$.

Definition

The set of eigenvalues of the operator A is called *a resonant* one if a relation of the form

$$\lambda_s = (m, \lambda) + ik,$$

is satisfied, where components of m are integer non-negative, $|m| \geq 2$, k is integer. This relation is called *a resonance relation* or just *a resonance*. The value $|m|$ is called *an order of the resonance*.

Note that number of resonances of given order $|m|$ is finite.

Assume that eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the operator A are all different. So, the the eigenvectors $e_1, e_2, \ldots, e_n$ of the complexified operator A form a basis in $\mathbb{C}^n$.

Let some system $\mathcal{S}$ of resonance relations be given. We will assume that $\mathcal{S}$ contains all resonance relations which can be derived from any subsystem of $\mathcal{S}$.

Definition

A vector monomial $e^{ikt}x^m e_s$ is called a *resonant* one for resonances in $\mathcal{S}$ if the resonance relation $\lambda_s = (m, \lambda) + ik$ is presented in the system $\mathcal{S}$.

Definition

A system

$$\dot{x} = Ax + \ldots$$

is said to be in the resonant normal form for resonances from $\mathcal{S}$ if the nonlinear part of its right hand side is a sum of resonant vector monomials.

Reduction to resonant normal form near periodic trajectory

Theorem

If eigenvalues of an equilibrium of time-periodic system do not satisfy resonance relations up to an order N inclusively except, may be, resonances from $\mathcal{S}$, then by a polynomial in space coordinates and periodic in time real close to the identical transformation of variables

$$x = y + O(|y|^2)$$

the system is reducible to the form

$$\dot{y} = Ay + w(y,t) + O(|y|^{N+1})$$

were w is a sum of resonant vector monomials of degrees not exceeding N.

Thus, the system without the term $O(|y|^{N+1})$ (also called *a truncated system*) is in a resonant normal form.

Corollary

If there are no resonances of any order, except, may be, resonances from $\mathcal{S}$, then a formal transformation of variables reduces the original system to a system in a formal resonant normal form.

Example: normal form for Neimark-Sacker bifurcation

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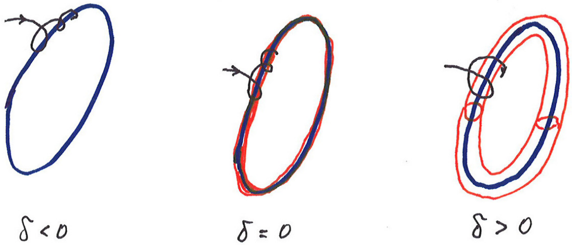

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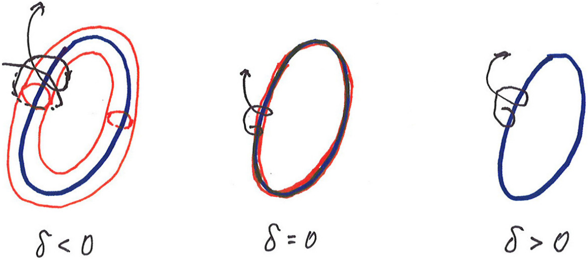

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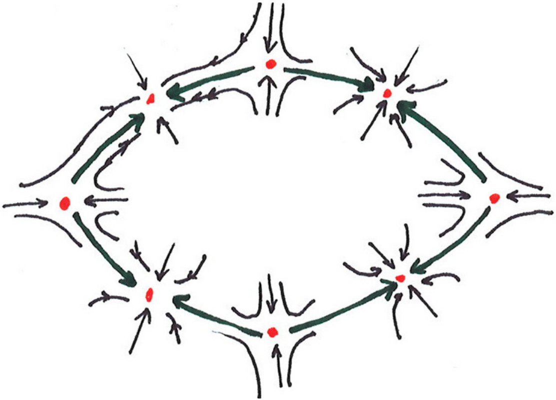

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On stability loss of periodic trajectory near the resonance 1:3

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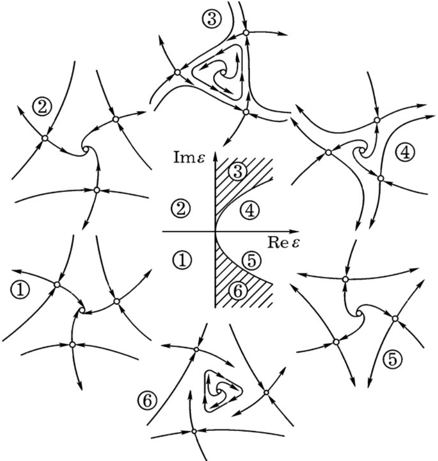

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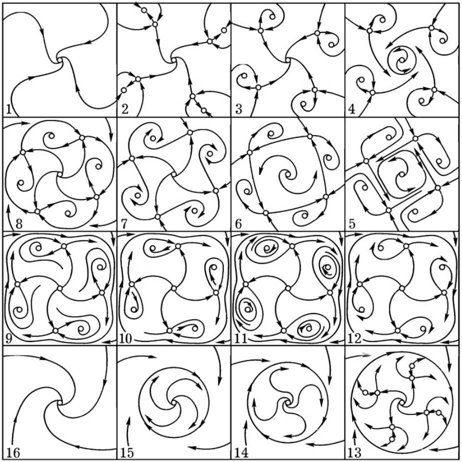
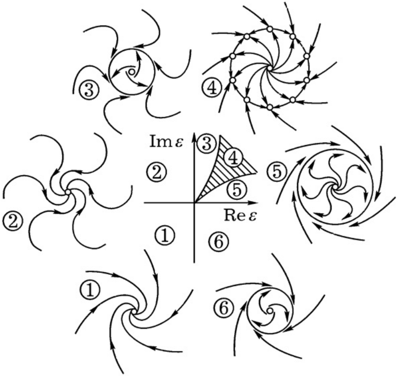


What happens if a pair of complex-conjugated multipliers ρ, $\bar{\rho}$ cross the unit circle near the points $e^{\pm\frac{2\pi ki}{q}}$, $q \geq 5$, k and q are co-prime?
The bifurcation diagrams are all similar, here is the diagram for $q = 5$, $k = 1$, V.I.Arnold, Geometrical methods in the theory of ordinary differential equations", $\varepsilon = \ln \rho - \frac{2\pi i}{5}$.



On period-doubling bifurcation for periodic trajectory

The period-doubling bifurcation is a local bifurcation which takes place in generic ODEs and maps when a periodic trajectory (or a fixed point, for a map) loses stability as a real multiplier crosses the unit circle in the complex plane in the point -1 .

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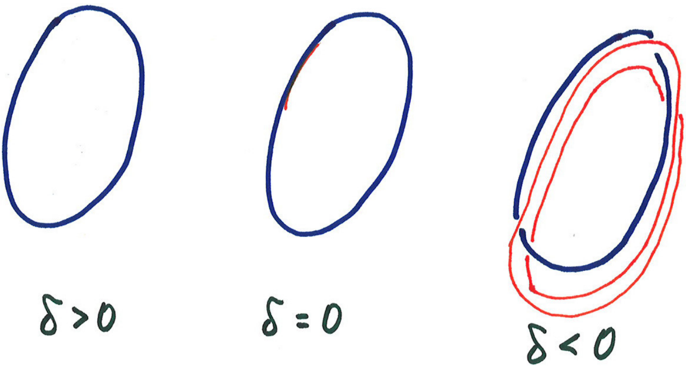
Here is the bifurcation diagram for ODEs for the supercritical case, $\rho = -1 + \delta$.



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Let α be a parameter of the family. For $\alpha \in (\alpha_1, \alpha_2)$ there is a stable periodic trajectory of a period T. At $\alpha = \alpha_2$ a real multiplier of this trajectory passes through -1, the trajectory loses its stability, and a new stable periodic trajectory of the period $2T$ branches off. This trajectory remains stable for $\alpha \in (\alpha_2, \alpha_3)$. At $\alpha = \alpha_3$ a real multiplier of this trajectory passes through -1, the trajectory loses its stablity, and a new stable periodic trajectory of the period $4T$ branches off, and so on. For $\alpha \in (\alpha_n, \alpha_{n+1})$ there is a a stable periodic trajectory of the period $2^n T$. The sequence $\{\alpha_n\}$ has a limit α_* as $n \to \infty$.

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Moreover, the distance between successive moments of bifurcation decay about as in a geometric progression with *universal* common ratio:

$$\lim_{k\to\infty} \frac{\alpha_k - \alpha_{k-1}}{\alpha_{k+1} - \alpha_k} = \mu_F = 4.6692\ldots$$

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The discussion of explanation of this phenomenon is postponed till the section about period doubling for maps.

This phenomenon is called *Feigenbaum's universality in period-doubling cascade*. The constant μ_F is called *the Feigenbaum constant*. It is a new mathematical constant like e or π.

Consider a map

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Resonant monomials, resonant normal form near fixed point

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Reduction to resonant normal form near fixed point

Procedure of reduction to resonant normal form near fixed point is analogous to that near an equilibrium

The map under consideration has the form

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where $v_r(x)$ is the homogeneous vector polynomial of x of degree r.

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We are looking for a transformation of variables $x \mapsto y$ of the form

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where $h_r(y), w_r(y)$ are homogeneous vector polynomials of y of degree r, and $w_r(y)$ contains only resonant monomials.

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which reduces the map to the form

$$y \mapsto Ay + w(y) + O(|y|^{N+1}), \;\; w(y) = w_2(y) + w_3(y) + \ldots + w_N(y)$$

where $h_r(y), w_r(y)$ are homogeneous vector polynomials of y of degree r, and $w_r(y)$ contains only resonant monomials. Plugging the transformation of variables into original map, assuming that the transformed map has required form and equating terms of order r we get *a homological equation*

$$h_r(Ay) - Ah_r(y) = V_r(y) - w_r(y)$$

where V_r is the homogeneous vector polynomial of degree r whose coefficients are expressed through coefficients of $v_2, \ldots, v_r,\ h_2, \ldots, h_{r-1},\ w_2, \ldots, w_{r-1}$.

Reduction to resonant normal form near fixed point

Procedure of reduction to resonant normal form near fixed point is analogous to that near an equilibrium

The map under consideration has the form

$$x \mapsto Ax + V(x),\ \ V(x) = v_2(x) + v_3(x) + \ldots + v_N(x) + O(|x|^{N+1})$$

where $v_r(x)$ is the homogeneous vector polynomial of x of degree r.

We are looking for a transformation of variables $x \mapsto y$ of the form

$$x = y + h(y),\ \ h(y) = h_2(y) + h_3(y) + \ldots + h_N(y)$$

which reduces the map to the form

$$y \mapsto Ay + w(y) + O(|y|^{N+1}),\ \ w(y) = w_2(y) + w_3(y) + \ldots + w_N(y)$$

where $h_r(y), w_r(y)$ are homogeneous vector polynomials of y of degree r, and $w_r(y)$ contains only resonant monomials. Plugging the transformation of variables into original map, assuming that the transformed map has required form and equating terms of order r we get *a homological equation*

$$h_r(Ay) - Ah_r(y) = V_r(y) - w_r(y)$$

where V_r is the homogeneous vector polynomial of degree r whose coefficients are expressed through coefficients of $v_2, \ldots, v_r,\ h_2, \ldots, h_{r-1},\ w_2, \ldots, w_{r-1}$.

Take as $w_r(y,t)$ the sum of resonant monomials in $V_r(y,t)$.

Lemma

For this choice of w_r the homological equation has a solution h_r in the form of a sum of non-resonant monomials. The solution in such form is a unique.

Proof.

Let $e_1,\ e_2, \dots, e_n$ be eigenvectors of the complexified operator A, that correspond to the eigenvalues $\lambda_1,\ \lambda_2, \dots, \lambda_n$. The eigenvalues of A are all different, and so the eigenvectors form a basis in $\mathbb{C}^n$. Let $y_1, y_2, \dots, y_n$ be coordinates of y in this basis. Denote $U_r(y) = V_r(y) - w_r(y)$. Then

$$U_r = \sum_{s=1,\dots,n;\ |m|=r} U_{s,m} y^m e_s,\ \ h = \sum_{s=1,\dots,n;\ |m|=r} h_{s,m} y^m e_s$$

Equating in the homological equation the coefficients in front of $y^m e_s$, we get

$$(\lambda^m - \lambda_s) h_{s,m} = U_{s,m}$$

Thus, $h_{s,m} = U_{s,m}/(\lambda^m - \lambda_s)$. If y is real, then $h(y)$ is real. This completes the proof. □